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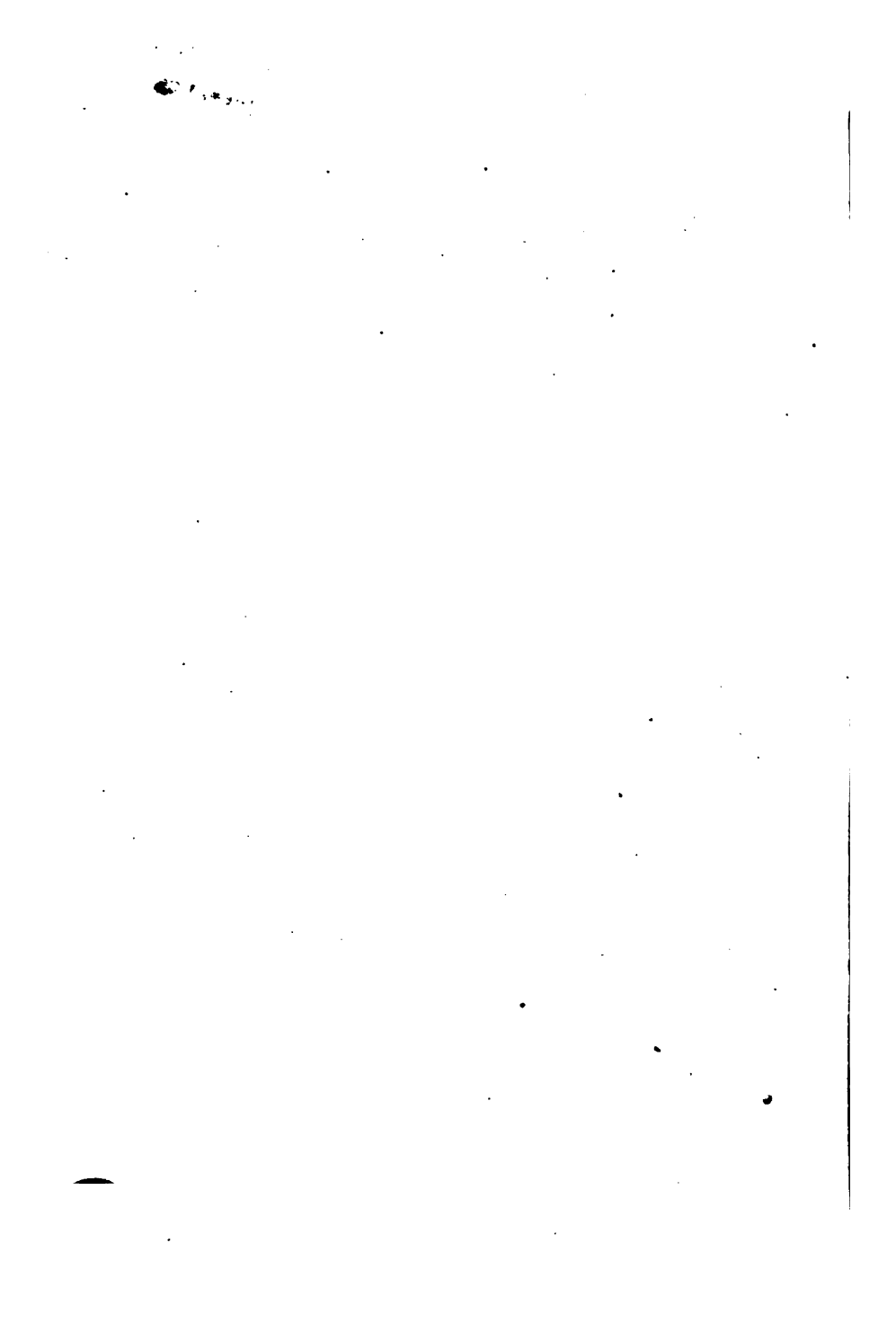
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Mrs. Angel
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Feb. 18th. 1867.

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OUTLINES
OF
MATHEMATICAL SCIENCE
101637

FOR THE SCHOOL-ROOM.

BY
CHARLES DAVIES, LL.D.,
AUTHOR OF A FULL COURSE OF MATHEMATICS.

NEW YORK:
PUBLISHED BY A. S. BARNES & CO.,
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1867.

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PREFACE AND PLAN.



THE object of this work is to furnish to the teacher, in the school-room, an aid and guide in his daily labors of giving Mathematical Instruction.

The work is but an analysis, in an abridged form, of that system of mathematical instruction which has been steadily pursued at the Military Academy over a third of a century, and which has given to that institution its celebrity as a school of mathematical science.

It is the essence of this system that a principle be taught before it is applied to practice—that Science should precede Art—and that general laws and general principles are the only true foundations of knowledge. Where, then, for the instruction of the young, can we find these laws and principles?

The first impressions which the child receives of Number and Space are the foundations of his mathematical knowledge. They form, as it were, a part of his intellectual being. The laws of mathematical science are generalized truths derived from the con-

sideration of Number and Space. All the processes of inquiry and investigation are conducted according to fixed laws, and form a science; and every new thought and higher impression form additional links in the lengthening chain. Beginning with first principles and elementary combinations, and guided by simple laws, he will go forward from the exercises of Mental Arithmetic to the higher analysis of Mathematical Science on an ascent so gentle, and with a progress so steady, as scarcely to note the changes. And, indeed, why should he? For all mathematical processes are alike in their nature—are governed by the same laws, exercise the same faculties, and lift the mind towards the same eminence.

The educator regards mathematical science as the great means of accomplishing his work. The definitions present clear and separate ideas, which the mind readily apprehends. The axioms afford the simplest exercises of the reasoning faculty, and the most satisfactory results in the early use and employment of that faculty. The trains of reasoning which follow, are combinations, according to logical rules, of what has been previously fully comprehended; and the mind and the argument grow together, so that the thread of science and the warp of the intellect entwine themselves, and become inseparable. Such a training will lay the foundation of systematic

knowledge, so greatly preferable to conjectural judgments.

With these general views, and under a firm conviction that mathematical science must become the great basis of education, I have bestowed much time and labor on its analysis, as a subject of knowledge. I have endeavored to present its elements separately, and in their connections; to point out and note the mental faculties which it calls into exercise; to show why and how it develops those faculties; and in what respect it gives to the whole mental machinery greater power and certainty of action than can be attained by other studies.

Mathematical knowledge differs from every other kind of knowledge in this: it is, as it were, a web of connected principles spun out from a few abstract ideas, until it has become one of the great means of intellectual development and of practical utility. And if I am permitted to extend the figure, I may add, that the web of the spider, though perfectly simple, if we see the end and understand the way in which it is put together, is yet too complicated to be unravelled, unless we begin at the right point, and *observe the law of its formation*. So with mathematical science. It is evolved from a few—a very few—elementary and intuitive principles; the law of its

evolution is simple but exacting, and to begin at the right place and proceed in the right way, is all that is necessary to make the subject easy, interesting, and useful.

I have endeavored to point out the place of beginning, and to indicate the way to the mathematical student. I am aware that he is starting on a road where the guide-boards resemble each other, and where, for the want of careful observation, they are often mistaken; I have sought, therefore, to furnish him with the maps and guide-books of an old traveller.

The leading idea, in the construction of the work, has been, to afford substantial aid to the professional teacher. The nature of his duties—their inherent difficulties—the perplexities which meet him at every step—the want of sympathy and support in his hours of discouragement—(and they are many)—are circumstances which awaken a lively interest in the hearts of all who have shared the toils, and been themselves laborers in the same vineyard. He takes his place in the schoolhouse, by the roadside, and there, removed from the highways of life, spends his days in raising the feeble mind of childhood to strength—in planting aright the seeds of knowledge—in curbing the turbulence of passion—in eradicating evil

and inspiring good. The fruits of his labors are seen but once in a generation. The boy must grow to manhood and the girl become a matron before he is certain that his labors have not been in vain.

Yet, to the teacher is committed the high trust of forming the intellectual, and, to a certain extent, the moral development of a people. He holds in his hands the keys of knowledge. If the first moral impressions do not spring into life at his bidding, he is at the source of the stream, and gives direction to the current. Although himself imprisoned in the schoolhouse, his influence and his teachings affect all conditions of society, and reach over the whole horizon of civilization. He impresses himself on the young of the age in which he lives, and lives again in the age which succeeds him.

All good teaching must flow from copious knowledge. The shallow fountain cannot emit a vigorous stream. In the hope of doing something that may be useful to the professional teacher, I have attempted a careful and full analysis of mathematical science. I have spread out, in detail, those methods which have been carefully examined and subjected to the test of long experience. If they are the right methods, they will serve as standards of teaching; for, the principles of imparting instruction are the same for all branches of education.

The system which I have indicated is complete in itself. It lays open to the teacher the entire skeleton of the science—exhibits all its parts separately and in their connection. It explains a course of reasoning simple in itself, and applicable not only to every process in mathematical science, but to all processes of argumentation in every subject of knowledge.

The teacher who thus combines science with art, no longer regards Arithmetic as a mere treadmill of mechanical labor, but as a means—and the simplest means—of teaching the art and science of reasoning on quantity; and this is the logic of mathematics. If he would accomplish well his work, he must so instruct his pupils that they shall apprehend clearly, think quickly and correctly, reason justly, and open their minds freely to the reception of all truth.

It may be proper to remark, that this work is an abridgment, with some changes and modifications, of a larger work entitled, "Logic and Utility of Mathematics."

FISHKILL-ON-HUDSON,
January, 1867.

CONTENTS.

SECTION I.

LOGIC OF MATHEMATICS.

	SECTION
Definitions—Different Kinds.....	1-6
Rules for Constructing Definitions.....	5
Operations of the Mind concerned in Reasoning.....	6-10
Apprehension.....	7
Judgment.....	8
Reasoning, or Discourse.....	9
Language.....	10, 11
Abstraction.....	12
Different Senses of Abstraction.....	13
Generalization.....	14
Singular Terms—Common Terms.....	15
Classification.....	16-19
Nature of Common Terms.....	20
Science.....	21
Art.....	22
Knowledge.....	23
Facts and Truths.....	24
Intuitive Truths.....	25
Logic or Reasoning.....	26
Induction.....	27-29
Deduction.....	30

Propositions—Different Kinds.....	SECTION 31-35
Syllogism.....	36-37

SECTION II.

OUTLINES OF MATHEMATICAL SCIENCE.

Mathematics Defined.....	38
Quantity Defined.....	39
Number—Different Kinds.....	40-43
Language of Number.....	44
Space Defined.....	45, 46
Portions of Space.....	47-50
Parts of Mathematics.....	51
Kinds of Number.....	52
Language of Mathematics.....	53
Its Symbolical Language.....	54
Symbols denoting Quantities.....	55
Arithmetic Defined.....	56
Geometry.....	57
Algebra.....	58
Analysis.....	59
Different Branches of Mathematics.....	60
Signs of Operation.....	61
Remarks on Language.....	62-66

SECTION III.

SCIENCE OF NUMBERS.

First Notions of Numbers.....	67
Three Forms of Language.....	68, 69

	SECTION
Ideas of Numbers Generalized.....	70, 71
Unity and a Unit Defined.....	72
Alphabet Defined.....	73
Arithmetical Alphabet.....	74
Spelling and Reading in Addition.....	75
General Reading.....	76-79
Spelling and Reading in Subtraction.....	80, 81
Spelling and Reading in Multiplication.....	82
Spelling and Reading in Division.....	83
Units increasing by Scale of Tens.....	84-88
Scales—Uniform and Varying.....	89
Units increasing by Scale of Tens.....	90, 91
Units increasing by Varying Scales.....	92, 93
Integral Units—Different Kinds.....	94, 95
Abstract Units Defined.....	96, 97
Units of Currency.....	98-100
Units of Weight.....	101-104
Units of Time.....	105
Units of Length.....	106
Units of Surface.....	107-110
Units of Volume.....	111-114
Units of Angular Measure.....	115
Advantages of System of Unities.....	116
Application to the Four Ground-Rules.....	117-119
Logical Form for Addition.....	120
Logical Form for Subtraction.....	121
Forms for Multiplication and Division.....	122-123

METRIC SYSTEM.

Naming—Measurement of Length.....	124, 125
Square Measure.....	126
Measures of Volumes.....	127
Dry Measure.....	128

	SECTION
Liquid Measure.....	129
Weights.....	130
Nature of Metric System	131

FRACTIONAL UNITS.

Scale of Tens.....	132-134
Fractional Units in General.....	135-137
Advantages of Fractional Units.....	138, 139

RATIO AND PROPORTION.

Ratio Defined, and Properties	140-144
-------------------------------------	---------

SECTION IV.

GEOMETRY.

Geometry—Defined, Demonstration.....	145
Geometrical Magnitudes.....	146
Lines—Straight and Curved	147
Surfaces—Plain and Curved	148
Plane Figures—Different Kinds.....	149-153
Volumes—Different Kinds.....	154
Angles.....	155
General Remarks	156-157
Comparison of Figures	158-164
Properties of Figures	165
Marks of what may be Proved.....	166
Demonstration—Its Nature.....	167-170
Two Kinds of Demonstration	171-175
Proportion of Figures.....	176

	SECTION
Proportion Defined.....	177
Inverse Proportion.....	178
Comparison of Figures.....	179
Comparison of Volumes.....	180
Ratio of Two Magnitudes.....	181
Recapitulation for Geometry.....	182
Suggestions to those who Teach Geometry.....	182

SECTION V.

ANALYSIS.

Analysis Defined.....	188
Its General Principles.....	188-187
Principal branches of Analysis.....	188
Algebra.....	189
Analytical Geometry.....	190
Analytical Trigonometry.....	191
Differential and Integral Calculus.....	192-194

SECTION VI.

ALGEBRA.

General Properties.....	195-196
How Quantities may be Increased or Diminished.....	197
Signs Denoting the Operations of Change.....	198
Symbols—their Nature.....	199
Coefficient—Its Use.....	200-201
Exponent—Its Use.....	202-208

	SECTION
Liquid Measure.....	129
Weights.....	130
Nature of Metric System.....	131

FRACTIONAL UNITS.

Scale of Tens.....	132-134
Fractional Units in General.....	135-137
Advantages of Fractional Units.....	138, 139

RATIO AND PROPORTION.

Ratio Defined, and Properties.....	140-144
------------------------------------	---------

SECTION IV.

GEOMETRY.

Geometry—Defined, Demonstration.....	145
Geometrical Magnitudes.....	146
Lines—Straight and Curved.....	147
Surfaces—Plain and Curved.....	148
Plane Figures—Different Kinds.....	149-153
Volumes—Different Kinds.....	154
Angles.....	155
General Remarks.....	156-157
Comparison of Figures.....	158-164
Properties of Figures.....	165
Marks of what may be Proved.....	166
Demonstration—Its Nature.....	167-170
Two Kinds of Demonstration.....	171-175
Proportion of Figures.....	176

	SECTION
Proportion Defined.....	177
Inverse Proportion.....	178
Comparison of Figures.....	179
Comparison of Volumes.....	180
Ratio of Two Magnitudes.....	181
Recapitulation for Geometry.....	182
Suggestions to those who Teach Geometry.....	182

SECTION V.

ANALYSIS.

Analysis Defined.....	188
Its General Principles.....	188-187
Principal branches of Analysis.....	188
Algebra.....	189
Analytical Geometry.....	190
Analytical Trigonometry.....	191
Differential and Integral Calculus.....	192-194

SECTION VI.

ALGEBRA.

General Properties.....	195-196
How Quantities may be Increased or Diminished.....	197
Signs Denoting the Operations of Change.....	198
Symbols—their Nature.....	199
Coefficient—Its Use.....	200-201
Exponent—Its Use.....	202-203

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CONTENTS.

SECTION I.

LOGIC OF MATHEMATICS.

	SECTION
Definitions—Different Kinds	1-6
Rules for Constructing Definitions	5
Operations of the Mind concerned in Reasoning	6-10
Apprehension	7
Judgment	8
Reasoning, or Discourse	9
Language	10, 11
Abstraction	12
Different Senses of Abstraction	13
Generalization	14
Singular Terms—Common Terms	15
Classification	16-19
Nature of Common Terms	20
Science	21
Art	22
Knowledge	23
Facts and Truths	24
Intuitive Truths	25
Logic or Reasoning	26
Induction	27-29
Deduction	30

Propositions—Different Kinds.....	SECTION 31-35
Syllogism.....	36-37

SECTION II.

OUTLINES OF MATHEMATICAL SCIENCE.

Mathematics Defined.....	38
Quantity Defined.....	39
Number—Different Kinds.....	40-43
Language of Number.....	44
Space Defined.....	45, 46
Portions of Space.....	47-50
Parts of Mathematics.....	51
Kinds of Number.....	52
Language of Mathematics.....	53
Its Symbolical Language.....	54
Symbols denoting Quantities.....	55
Arithmetic Defined.....	56
Geometry.....	57
Algebra.....	58
Analysis.....	59
Different Branches of Mathematics.....	60
Signs of Operation.....	61
Remarks on Language.....	62-66

SECTION III.

SCIENCE OF NUMBERS.

First Notions of Numbers.....	67
Three Forms of Language.....	68, 69

	SECTION
Ideas of Numbers Generalized.....	70, 71
Unity and a Unit Defined.....	72
Alphabet Defined.....	73
Arithmetical Alphabet.....	74
Spelling and Reading in Addition.....	75
General Reading.....	76-79
Spelling and Reading in Subtraction.....	80, 81
Spelling and Reading in Multiplication.....	82
Spelling and Reading in Division.....	83
Units increasing by Scale of Tens.....	84-88
Scales—Uniform and Varying.....	89
Units increasing by Scale of Tens.....	90, 91
Units increasing by Varying Scales.....	92, 93
Integral Units—Different Kinds.....	94, 95
Abstract Units Defined.....	96, 97
Units of Currency.....	98-100
Units of Weight.....	101-104
Units of Time.....	105
Units of Length.....	106
Units of Surface.....	107-110
Units of Volume.....	111-114
Units of Angular Measure.....	115
Advantages of System of Unities.....	116
Application to the Four Ground-Rules.....	117-119
Logical Form for Addition.....	120
Logical Form for Subtraction.....	121
Forms for Multiplication and Division.....	122-123

METRIC SYSTEM.

Naming—Measurement of Length.....	124, 125
Square Measure.....	126
Measures of Volumes.....	127
Dry Measure.....	128

	SECTION
Liquid Measure.....	129
Weights.....	180
Nature of Metric System.....	181

FRACTIONAL UNITS.

Scale of Tens.....	132-134
Fractional Units in General.....	135-137
Advantages of Fractional Units.....	138, 139

RATIO AND PROPORTION.

Ratio Defined, and Properties.....	140-144
------------------------------------	---------

SECTION IV.

GEOMETRY.

Geometry—Defined, Demonstration.....	145
Geometrical Magnitudes.....	146
Lines—Straight and Curved.....	147
Surfaces—Plain and Curved.....	148
Plane Figures—Different Kinds.....	149-153
Volumes—Different Kinds.....	154
Angles.....	155
General Remarks.....	156-157
Comparison of Figures.....	158-164
Properties of Figures.....	165
Marks of what may be Proved.....	166
Demonstration—Its Nature.....	167-170
Two Kinds of Demonstration.....	171-175
Proportion of Figures.....	176

	SECTION
Proportion Defined.....	177
Inverse Proportion.....	178
Comparison of Figures.....	179
Comparison of Volumes.....	180
Ratio of Two Magnitudes.....	181
Recapitulation for Geometry.....	182
Suggestions to those who Teach Geometry.....	182

SECTION V.

ANALYSIS.

Analysis Defined.....	188
Its General Principles.....	188-187
Principal branches of Analysis.....	188
Algebra.....	189
Analytical Geometry.....	190
Analytical Trigonometry.....	191
Differential and Integral Calculus.....	192-194

SECTION VI.

ALGEBRA.

General Properties.....	195-196
How Quantities may be Increased or Diminished.....	197
Signs Denoting the Operations of Change.....	198
Symbols—their Nature.....	199
Coefficient—Its Use.....	200-201
Exponent—Its Use.....	202-203

	SECTION
Division—Its Signs	204
Extraction of Roots—Signs	205
Minus Sign	206–207
Subtraction	208
Multiplication	209
Rules for the Signs	209–211
Zero and Infinity	212–216
Nature of the Equation	217–221
Axioms and Operations	221
General Remarks	223
Suggestions to those who Teach Algebra	225

SECTION I.

LOGIC OF MATHEMATICS.

DEFINITIONS—OPERATIONS OF THE MIND—TERMS—LOGIC—
INDUCTIVE—DEDUCTIVE—SYLLOGISM.

DEFINITIONS.

§ 1. DEFINITION is a metaphorical word, which literally signifies “laying down a boundary.” All definitions are of names, and of names only; but in some definitions, it is clearly apparent, that nothing is intended except to explain the meaning of the word; while in others, besides explaining the meaning of the word, it is also implied that there exists, or may exist, a *thing* corresponding to the word.

§ 2. Definitions which do not imply the existence of things corresponding to the words defined, are those usually found in the Dictionary of one’s own language. They explain only the meaning of the word or term, by giving some equivalent expression which may happen to be better known. Definitions

which imply the existence of things corresponding to the words defined, do more than this.

For example: "A triangle is a rectilinear figure having three sides." This definition does two things:

- 1st. It explains the meaning of the word triangle; and,
- 2d. It implies that there exists, or may exist, a rectilinear figure having three sides.

§ 3. To define a word when the definition is to imply the existence of a thing, is to select from all the properties of the thing those which are most simple, general, and obvious; and the properties must be very well known to us before we can decide which are the fittest for this purpose. Hence, a thing may have many properties besides those which are named in the definition of the word which stands for it.

§ 4. In Mathematics, and indeed in all exact sciences, names imply the existence of the things which they name; and the definitions of those names express attributes of the things; so that no correct definition whatever, of any mathematical term, can be devised, which shall not express certain attributes of the thing corresponding to the name. Every definition of this class is a tacit assumption of some proposition which is expressed by means of the definition, and which gives to such definition its importance.

§ 5. All the reasonings in mathematics, which rest ultimately on definitions, do, in fact, rest on the intuitive inference, that things corresponding to the words defined have a *conceivable* existence as subjects of thought, and do or may have *proximately*, an *actual* existence.*

OPERATIONS OF THE MIND CONCERNED IN REASONING.

§ 6. There are three operations of the mind which are immediately concerned in reasoning.

1st. Simple Apprehension; 2d. Judgment; 3d. Reasoning or Discourse.

§ 7. Simple apprehension is the notion (or conception) of an object in the *mind*, analogous to the perception of the senses. It is either Incomplex or Complex. Incomplex Apprehension is of one object, or of several without any *relation* being perceived between them, as of a triangle, a square, or a circle: Complex is of several with such a *relation*, as of a triangle within a circle, or a circle within a square.

§ 8. Judgment is the comparing together in the mind two of the notions (or ideas) which are the ob-

* There are four rules which aid us in framing definitions.

1st. The definition must be *adequate*: that is, neither too extended, nor too narrow for the word defined.

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CONTENTS.

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Definitions—Different Kinds.....	1-6
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Apprehension.....	7
Judgment.....	8
Reasoning, or Discourse.....	9
Language.....	10, 11
Abstraction.....	12
Different Senses of Abstraction.....	13
Generalization.....	14
Singular Terms—Common Terms.....	15
Classification.....	16-19
Nature of Common Terms.....	20
Science.....	21
Art.....	22
Knowledge.....	23
Facts and Truths.....	24
Intuitive Truths.....	25
Logic or Reasoning.....	26
Induction.....	27-29
Deduction.....	30

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The teacher who thus combines science with art, no longer regards Arithmetic as a mere treadmill of mechanical labor, but as a means—and the simplest means—of teaching the art and science of reasoning on quantity; and this is the logic of mathematics. If he would accomplish well his work, he must so instruct his pupils that they shall apprehend clearly, think quickly and correctly, reason justly, and open their minds freely to the reception of all truth.

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FISHKILL-ON-HUDSON,
January, 1867.

CONTENTS.

SECTION I.

LOGIC OF MATHEMATICS.

	SECTION
Definitions—Different Kinds.....	1-6
Rules for Constructing Definitions.....	5
Operations of the Mind concerned in Reasoning.....	6-10
Apprehension.....	7
Judgment.....	8
Reasoning, or Discourse.....	9
Language.....	10, 11
Abstraction.....	12
Different Senses of Abstraction.....	13
Generalization.....	14
Singular Terms—Common Terms.....	15
Classification.....	16-19
Nature of Common Terms.....	20
Science.....	21
Art.....	22
Knowledge.....	23
Facts and Truths.....	24
Intuitive Truths.....	25
Logic or Reasoning.....	26
Induction.....	27-29
Deduction.....	30

	SECTION
Propositions—Different Kinds.....	81-85
Syllogism.....	86-87

SECTION II.

OUTLINES OF MATHEMATICAL SCIENCE.

Mathematics Defined.....	38
Quantity Defined.....	39
Number—Different Kinds.....	40-43
Language of Number.....	44
Space Defined.....	45, 46
Portions of Space.....	47-50
Parts of Mathematics.....	51
Kinds of Number.....	52
Language of Mathematics.....	53
Its Symbolical Language.....	54
Symbols denoting Quantities.....	55
Arithmetic Defined.....	56
Geometry.....	57
Algebra.....	58
Analysis.....	59
Different Branches of Mathematics.....	60
Signs of Operation.....	61
Remarks on Language.....	62-66

SECTION III.

SCIENCE OF NUMBERS.

First Notions of Numbers.....	67
Three Forms of Language.....	68, 69

	SECTION
Ideas of Numbers Generalized.....	70, 71
Unity and a Unit Defined.....	72
Alphabet Defined.....	73
Arithmetical Alphabet.....	74
Spelling and Reading in Addition.....	75
General Reading.....	76-79
Spelling and Reading in Subtraction.....	80, 81
Spelling and Reading in Multiplication.....	82
Spelling and Reading in Division.....	83
Units increasing by Scale of Tens.....	84-88
Scales—Uniform and Varying.....	89
Units increasing by Scale of Tens.....	90, 91
Units increasing by Varying Scales.....	92, 93
Integral Units—Different Kinds.....	94, 95
Abstract Units Defined.....	96, 97
Units of Currency.....	98-100
Units of Weight.....	101-104
Units of Time.....	105
Units of Length.....	106
Units of Surface.....	107-110
Units of Volume.....	111-114
Units of Angular Measure.....	115
Advantages of System of Unities.....	116
Application to the Four Ground-Rules.....	117-119
Logical Form for Addition.....	120
Logical Form for Subtraction.....	121
Forms for Multiplication and Division.....	122-123

METRIC SYSTEM.

Naming—Measurement of Length.....	124, 125
Square Measure.....	126
Measures of Volumes.....	127
Dry Measure.....	128

	SECTION
Liquid Measure.....	129
Weights.....	130
Nature of Metric System.....	131

FRACTIONAL UNITS.

Scale of Tens.....	132-134
Fractional Units in General.....	135-137
Advantages of Fractional Units.....	138, 139

RATIO AND PROPORTION.

Ratio Defined, and Properties.....	140-144
------------------------------------	---------

SECTION IV.

GEOMETRY.

Geometry—Defined, Demonstration.....	145
Geometrical Magnitudes.....	146
Lines—Straight and Curved.....	147
Surfaces—Plain and Curved.....	148
Plane Figures—Different Kinds.....	149-153
Volumes—Different Kinds.....	154
Angles.....	155
General Remarks.....	156-157
Comparison of Figures.....	158-164
Properties of Figures.....	165
Marks of what may be Proved.....	166
Demonstration—Its Nature.....	167-170
Two Kinds of Demonstration.....	171-175
Proportion of Figures.....	176

	SECTION
Proportion Defined.....	177
Inverse Proportion.....	178
Comparison of Figures.....	179
Comparison of Volumes.....	180
Ratio of Two Magnitudes.....	181
Recapitulation for Geometry.....	182
Suggestions to those who Teach Geometry.....	182

SECTION V.

ANALYSIS.

Analysis Defined.....	188
Its General Principles.....	188-187
Principal branches of Analysis.....	188
Algebra.....	189
Analytical Geometry.....	190
Analytical Trigonometry.....	191
Differential and Integral Calculus.....	192-194

SECTION VI.

ALGEBRA.

General Properties.....	195-196
How Quantities may be Increased or Diminished.....	197
Signs Denoting the Operations of Change.....	198
Symbols—their Nature.....	199
Coefficient—Its Use.....	200-201
Exponent—Its Use.....	202-203

	SECTION
Division—Its Signs	204
Extraction of Roots—Signs	205
Minus Sign	206–207
Subtraction	208
Multiplication	209
Rules for the Signs	209–211
Zero and Infinity	212–216
Nature of the Equation	217–221
Axioms and Operations	221
General Remarks	223
Suggestions to those who Teach Algebra	225

SECTION I.

LOGIC OF MATHEMATICS.

DEFINITIONS—OPERATIONS OF THE MIND—TERMS—LOGIC—
INDUCTIVE—DEDUCTIVE—SYLLOGISM.

DEFINITIONS.

§ 1. DEFINITION is a metaphorical word, which literally signifies “laying down a boundary.” All definitions are of names, and of names only; but in some definitions, it is clearly apparent, that nothing is intended except to explain the meaning of the word; while in others, besides explaining the meaning of the word, it is also implied that there exists, or may exist, a *thing* corresponding to the word.

§ 2. Definitions which do not imply the existence of things corresponding to the words defined, are those usually found in the Dictionary of one’s own language. They explain only the meaning of the word or term, by giving some equivalent expression which may happen to be better known. Definitions

evolution is simple but exacting, and to begin at the right place and proceed in the right way, is all that is necessary to make the subject easy, interesting, and useful.

I have endeavored to point out the place of beginning, and to indicate the way to the mathematical student. I am aware that he is starting on a road where the guide-boards resemble each other, and where, for the want of careful observation, they are often mistaken; I have sought, therefore, to furnish him with the maps and guide-books of an old traveller.

The leading idea, in the construction of the work, has been, to afford substantial aid to the professional teacher. The nature of his duties—their inherent difficulties—the perplexities which meet him at every step—the want of sympathy and support in his hours of discouragement—(and they are many)—are circumstances which awaken a lively interest in the hearts of all who have shared the toils, and been themselves laborers in the same vineyard. He takes his place in the schoolhouse, by the roadside, and there, removed from the highways of life, spends his days in raising the feeble mind of childhood to strength—in planting aright the seeds of knowledge—in curbing the turbulence of passion—in eradicating evil

and inspiring good. The fruits of his labors are seen but once in a generation. The boy must grow to manhood and the girl become a matron before he is certain that his labors have not been in vain.

Yet, to the teacher is committed the high trust of forming the intellectual, and, to a certain extent, the moral development of a people. He holds in his hands the keys of knowledge. If the first moral impressions do not spring into life at his bidding, he is at the source of the stream, and gives direction to the current. Although himself imprisoned in the schoolhouse, his influence and his teachings affect all conditions of society, and reach over the whole horizon of civilization. He impresses himself on the young of the age in which he lives, and lives again in the age which succeeds him.

All good teaching must flow from copious knowledge. The shallow fountain cannot emit a vigorous stream. In the hope of doing something that may be useful to the professional teacher, I have attempted a careful and full analysis of mathematical science. I have spread out, in detail, those methods which have been carefully examined and subjected to the test of long experience. If they are the right methods, they will serve as standards of teaching; for, the principles of imparting instruction are the same for all branches of education.

The system which I have indicated is complete in itself. It lays open to the teacher the entire skeleton of the science—exhibits all its parts separately and in their connection. It explains a course of reasoning simple in itself, and applicable not only to every process in mathematical science, but to all processes of argumentation in every subject of knowledge.

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CONTENTS.

SECTION I.

LOGIC OF MATHEMATICS.

	SECTION
Definitions—Different Kinds	1-6
Rules for Constructing Definitions	5
Operations of the Mind concerned in Reasoning	6-10
Apprehension	7
Judgment	8
Reasoning, or Discourse	9
Language	10, 11
Abstraction	12
Different Senses of Abstraction	13
Generalization	14
Singular Terms—Common Terms	15
Classification	16-19
Nature of Common Terms	20
Science	21
Art	22
Knowledge	23
Facts and Truths	24
Intuitive Truths	25
Logic or Reasoning	26
Induction	27-29
Deduction	30

Propositions—Different Kinds.....	SECTION 81-85
Syllogism.....	86-87

SECTION II.

OUTLINES OF MATHEMATICAL SCIENCE.

Mathematics Defined.....	38
Quantity Defined.....	39
Number—Different Kinds.....	40-43
Language of Number.....	44
Space Defined.....	45, 46
Portions of Space.....	47-50
Parts of Mathematics.....	51
Kinds of Number.....	52
Language of Mathematics.....	53
Its Symbolical Language.....	54
Symbols denoting Quantities.....	55
Arithmetic Defined.....	56
Geometry.....	57
Algebra.....	58
Analysis.....	59
Different Branches of Mathematics.....	60
Signs of Operation.....	61
Remarks on Language.....	62-66

SECTION III.

SCIENCE OF NUMBERS.

First Notions of Numbers.....	67
Three Forms of Language.....	68, 69

	SECTION
Ideas of Numbers Generalized.....	70, 71
Unity and a Unit Defined.....	72
Alphabet Defined.....	73
Arithmetical Alphabet.....	74
Spelling and Reading in Addition.....	75
General Reading.....	76-79
Spelling and Reading in Subtraction.....	80, 81
Spelling and Reading in Multiplication.....	82
Spelling and Reading in Division.....	83
Units increasing by Scale of Tens.....	84-88
Scales—Uniform and Varying.....	89
Units increasing by Scale of Tens.....	90, 91
Units increasing by Varying Scales.....	92, 93
Integral Units—Different Kinds.....	94, 95
Abstract Units Defined.....	96, 97
Units of Currency.....	98-100
Units of Weight.....	101-104
Units of Time.....	105
Units of Length.....	106
Units of Surface.....	107-110
Units of Volume.....	111-114
Units of Angular Measure.....	115
Advantages of System of Unities.....	116
Application to the Four Ground-Rules.....	117-119
Logical Form for Addition.....	120
Logical Form for Subtraction.....	121
Forms for Multiplication and Division.....	122-123

METRIC SYSTEM.

Naming—Measurement of Length.....	124, 125
Square Measure.....	126
Measures of Volumes.....	127
Dry Measure.....	128

	SECTION
Liquid Measure.....	129
Weights.....	180
Nature of Metric System.....	181

FRACTIONAL UNITS.

Scale of Tens.....	132-134
Fractional Units in General.....	135-137
Advantages of Fractional Units.....	138, 139

RATIO AND PROPORTION.

Ratio Defined, and Properties.....	140-144
------------------------------------	---------

SECTION IV.

GEOMETRY.

Geometry—Defined, Demonstration.....	145
Geometrical Magnitudes.....	146
Lines—Straight and Curved.....	147
Surfaces—Plain and Curved.....	148
Plane Figures—Different Kinds.....	149-153
Volumes—Different Kinds.....	154
Angles.....	155
General Remarks.....	156-157
Comparison of Figures.....	158-164
Properties of Figures.....	165
Marks of what may be Proved.....	166
Demonstration—Its Nature.....	167-170
Two Kinds of Demonstration.....	171-175
Proportion of Figures.....	176

	SECTION
Proportion Defined.....	177
Inverse Proportion.....	178
Comparison of Figures.....	179
Comparison of Volumes.....	180
Ratio of Two Magnitudes.....	181
Recapitulation for Geometry.....	182
Suggestions to those who Teach Geometry.....	182

SECTION V.

ANALYSIS.

Analysis Defined.....	183
Its General Principles.....	183-187
Principal branches of Analysis.....	188
Algebra.....	189
Analytical Geometry.....	190
Analytical Trigonometry.....	191
Differential and Integral Calculus.....	192-194

SECTION VI.

ALGEBRA.

General Properties.....	195-196
How Quantities may be Increased or Diminished.....	197
Signs Denoting the Operations of Change.....	198
Symbols—their Nature.....	199
Coefficient—Its Use.....	200-201
Exponent—Its Use.....	202-208

	SECTION
Division—Its Signs	204
Extraction of Roots—Signs	205
Minus Sign	206–207
Subtraction	208
Multiplication	209
Rules for the Signs	209–211
Zero and Infinity	212–216
Nature of the Equation	217–221
Axioms and Operations	221
General Remarks	223
Suggestions to those who Teach Algebra	225

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QUANTITY

NUMBER.....

- 1. ABSTRACT.
- 2. CURRENCY.
- 3. WEIGHT.
- 4. TIME.

SPACE.....

- 1. LINES.
- 2. SURFACES.
- 3. VOLUMES.
- 4. ANGLES.

SECTION II.

OUTLINE OF MATHEMATICAL SCIENCE.

§ 38. **MATHEMATICS** is the science of quantity ; that is, the science which treats of the measure of quantity and the relations of quantities to each other.

Mathematics may be divided into two parts, Pure and Mixed.

The Pure Mathematics embraces the principles of the science, and all explanations of the processes by which those principles are derived from the laws of the abstract quantities, Number and Space.

The Mixed Mathematics embraces the applications of those principles to all investigations, and to the solution of all questions of a practical nature, whether they relate to abstract or concrete quantity.

§ 39. **QUANTITY** is any thing which can be increased, diminished, and measured. There are but two kinds of quantity : Number and Space.

NUMBER.

§ 40. A NUMBER is a unit or a collection of units. How do we apprehend the nature of Numbers?

First, by presenting to the mind, through the eye, a single thing, and calling it ONE. Then, presenting two things, and naming them two: then three things, and naming them THREE; and so on for other numbers. Thus, we acquire primarily, in a *concrete* form, our elementary notions of number, by perception, comparison, and reflection: for, we must first perceive how many things are numbered; then compare what is designated by the word one, with what is designated by the words two, three, etc., and then reflect on the results of such comparisons until we clearly apprehend the difference in the signification of the words. Having thus acquired, in a concrete form, our conceptions of numbers, we can consider numbers as separated from any particular objects, and thus form a conception of them in the abstract.

§ 41. The UNIT of a number is the single thing which forms the base of the number.

§ 42. An ABSTRACT NUMBER is one whose unit is not named; as, one, two, three, &c.

§ 43. A DENOMINATE NUMBER is one whose unit is named ; as, one foot, one pound, one day, &c. A denominate number which carries with it the idea of matter, is called a *concrete* number.

Hence, there are three kinds of numbers: Abstract, Denominate, and Concrete.

Five, is an abstract number—Five yards a denominate number, and five pounds a concrete number. A concrete number is always a denominate number.

LANGUAGE OF NUMBER.

§ 44. The language of number is mixed. It is composed partly of words taken from our common language, but its more perfect and comprehensive form is found in the various combinations of the ten characters called figures. These ten characters are the alphabet of this language, and the various ways in which they are combined will be fully explained under the head Arithmetic, a branch of Mathematics which treats of the properties of numbers, and of their language and their laws.

SPACE.

§ 45. SPACE is indefinite extension. We acquire our ideas of it by observing that *parts* of it are occupied by matter or bodies. This enables us to attach a definite idea to the word *place*. We are then able to say,

intelligibly, that a point is that which has place, or position in space, without occupying any part of it. Having conceived a second point in space, we can understand the important axiom, "A straight line is the shortest distance between two points;" and this line we call *length*, or a *dimension* of space.

§ 46. If we conceive a second straight line to be drawn, meeting the first, but lying in a direction directly from it, we shall have a second dimension of space, which we call *breadth*. If these lines be prolonged, in both directions, they will include four portions of space, which make up what is called a plane surface, or plane: hence, a plane has two dimensions, length and breadth. If now we draw a line on either side of this plane, we shall have another dimension of space, called *thickness*: hence, space has three dimensions—length, breadth, and thickness.

PORTIONS OF SPACE.

§ 47. A portion of space which has but one dimension, is called a *line*, and may be limited by two points, one at each extremity.

There are two kinds of lines, straight and curved. A straight line, is one which does not change its direction between any two of its points, but a curved line changes its direction at every point.

§ 48. A portion of space having two dimensions is called a surface. There are two kinds of surfaces—Plane Surfaces and Curved Surfaces. With the former, a straight line, having two points in common, will always coincide, however it may be placed, while with the latter it will not. The boundaries of surfaces are lines, straight or curved.

§ 49. A portion of space having three dimensions is called a Volume. Volumes are bounded either by plane or curved surfaces.

§ 50. A portion of space included between lines, or between planes, is called an angle. Hence, there are four kinds of geometrical magnitudes: viz. Lines, Surfaces, Volumes, and Angles.

PARTS OF MATHEMATICS.

§ 51. Referring to the diagram, on page 38, we have a complete map of Mathematical Science. We see,

1. That Quantity is divided into two kinds, Number and Space.

2. That number is divided into four kinds, Abstract Number, Currency, Weight, and Time.

3. That space is also divided into four kinds, Length, Surfaces, Volumes, and Angles.

4. All these quantities may be expressed by num-

bers. Of these numbers, one only is abstract and seven denominate—and of the seven denominate numbers, only one, weight, is concrete.

§ 52. The class, or kind of number, is determined by the Unit which forms its base: thus,

An abstract number has for its base the unit one.

A number in currency has for its base a unit of currency, as 1 dollar, 1 dime, 1 cent; or in the English currency, 1 £, 1 s., 1 d.; or in the French, 1 franc, or 1 Napoleon.

In weight, the unit is 1 pound, 1 hundred, 1 ton, or 1 ounce; or any other known unit.

In time, the unit is 1 day, 1 year, 1 week, 1 hour, &c.

In length, it is 1 foot, 1 yard, 1 rod, 1 mile, or any other known distance.

In surfaces, or square measure, the unit is 1 square inch, 1 square foot, 1 square yard, 1 acre, or any other known surface.

In volumes, the unit is 1 cubic foot, 1 cubic yard, 1 gallon, 1 quart, or any other known and limited portion of space.

In angles, the most common unit is the degree, though in geometry, we generally compare all angles with the right angle.

LANGUAGE OF MATHEMATICS.

§ 53. The language of Mathematics is mixed. Although composed mainly of symbols, which are defined with reference to the uses which are made of them, and therefore have a precise signification, it is also composed, in part, of words transferred from our common language. The symbols, although arbitrary signs, are, nevertheless, entirely general, as signs and instruments of thought; and when the sense in which they are used is once fixed, by definition, they preserve throughout the entire analysis precisely the same signification. The meaning of the words, borrowed from our common vocabulary, is often modified, and sometimes entirely changed, when the words are transferred to the language of science. They are then used in a particular sense, and are said to have a *technical* signification.

It is of the first importance that the elements of the language be clearly understood,—that the signification of every word or symbol be distinctly apprehended, and that the connection between the thought and the word or symbol which expresses it, be so well established that the one shall immediately suggest the other.

THE SYMBOLICAL LANGUAGE.

§ 54. The Symbolical language of Mathematics is divided into two parts:

1. The symbols which denote quantities ; and
2. The symbols which denote operations to be performed on quantities.

SYMBOLS DENOTING QUANTITIES.

§ 55. There are three classes of symbols which denote quantities :

- 1st. The ten characters, called figures :
- 2d. The straight line and curve ; and
- 3d. The letters of the alphabet.

ARITHMETIC.

§ 56. ARITHMETIC is that branch of Mathematics in which the quantities considered are represented by figures. Hence, it is the science of Numbers, when those numbers are represented by figures.

GEOMETRY.

§ 57. GEOMETRY is that branch of Mathematics which treats of the properties and relations of Lines, Surfaces, Volumes, and Angles, when represented by the pictorial language of which the straight line and curve are the elements. Hence, Geometry is the science of space, when space is represented by means of the straight line and curve.

ALGEBRA.

§ 58. ALGEBRA is that branch of Mathematics in which the quantities considered are numerical, and denoted by letters; and the operations to be performed on them, are indicated by signs.

ANALYSIS.

§ 59. ANALYSIS is a general term embracing all the operations which can be performed on quantities when represented by letters. In this branch of mathematics, all the quantities considered, whether of number or space, are represented by letters of the alphabet, and the operations to be performed on them are indicated by a few arbitrary signs.

Analysis, in its simplest form, takes the name of Algebra; Analytical Geometry, the Differential and Integral Calculus, extended to include the Theory of Variations, are its higher and most advanced branches.

THE BRANCHES OF MATHEMATICS DETERMINED BY
LANGUAGE.

§ 60. We have seen that when quantities are denoted by figures, the operations belong to Arithmetic; when represented by lines, to Geometry; and when denoted by letters, to Algebra. Hence, the divisions

of Mathematical Science into the subjects of Arithmetic, Geometry, and Algebra, arises entirely from the language employed. The methods of reasoning, and the laws governing the operations, are the same in all.

SIGNS OF OPERATION.

§ 61. The signs of operation are well known. They are named here, merely to preserve the symmetry in treating the subject. They are,

$+$, called plus, and denoting addition.

$-$, called minus, and denoting subtraction.

$\times$, called, the sign of multiplication.

$\div$, called, the sign of division.

$=$, called, the sign of equality.

$\sqrt{}$, called, the radical sign, or the sign of square root.

REMARKS ON MATHEMATICAL LANGUAGE.

§ 62. The alphabet of the Arithmetical language, as already shown, contains ten characters, called figures, each of which has a name, and when standing by itself denotes as many things as the name indicates. There are but three combinations of these characters—the first is formed by writing them in rows—the second by writing some of them over or under others—and the third, by using of the decimal point.

This language, having ten elements and three combinations, is more simple, more minute, and more exact than any other known form of expressing our thoughts. It records all the daily transactions of the world, involving number and quantity. The yearly income—the accumulations of property—the balance sheets of mercantile enterprise are all expressed in numbers, and may be written in figures. These ten little characters are not only the sleepless sentinels of trade and commerce, but they also make known all the practical results of scientific labor.

§ 63. The language of Geometry is pictorial, and has but two elements, the straight line and curve. The combinations of these simple elements give every form and variety of the geometrical language. Distance, surface, volume, and angle, are names denoting portions of space.

Under these four names every part of space, in form, extent, and dimension, is represented to the mind by means of the straight line and curve. This language is both simple and comprehensive. The shortest distance—the curve of grace and beauty—the smooth surface and the rugged boundary, are alike amenable to its laws. It presents to the mind, through the eye, the forms and relative magnitudes of all the heavenly bodies, and, also, of the most minute and delicate objects that are revealed by the

microscope. It is the connecting link between theoretical and practical knowledge in the mechanic arts, and the only language in which science speaks to labor. All the works of Architecture, Sculpture, and Painting, are but images of the imagination until they assume the geometrical forms.

§ 64. The language of analysis is more comprehensive than that of figures, or the pictorial language of Geometry; indeed, it embraces them both. Its elements are the leading and final letters of the alphabet, and a few arbitrary signs. The combinations of these elements are few in number and simple in form; and from these humble sources are derived the fruitful language of analytical science.

This language is minute, suggestive, certain, general, and comprehensive. It will express every property and relation of number—every form which the imagination has given to space—every moment of time which has elapsed since hours began to be numbered—and every motion which has taken place since matter began to move. One or the other of these three forms of mathematical language is in daily use in every part of the world, and especially so in every place where science is employed to guide the hand of labor—to investigate the laws of matter—or to enlarge the boundaries of knowledge.

§ 65. We have thus completed a very brief and general analytical view of Mathematical Science. We have endeavored to point out the character of the definitions, and the sources as well as the nature of the elementary and intuitive propositions on which the science rests; the kind of reasoning employed in its creation, and its divisions resulting from the use of different symbols, and differences of language. We shall now proceed to treat the subjects separately.

SECTION III.

SCIENCE OF NUMBERS.

FIRST NOTIONS OF NUMBERS.

§ 66. THERE is but a single elementary idea in the science of numbers: it is the idea of the UNIT ONE. There is but one way of impressing this idea on the mind. It is by presenting to the senses a single object; as, one apple, one peach, one pear, &c.

§ 67. There are three signs by means of which the idea of one is expressed and communicated. They are,

- 1st. The word ONE.
- 2d. The Roman character I.
- 3d. The figure 1.

The idea of one, may also be communicated through the ear, by the sound, one.

§ 68. If one be added to one, the idea thus arising is different from the idea of one, and is complex. This new idea has also three signs; viz. two, II., and 2.

If one be again added, that is, added to two, the new idea has likewise three signs; viz. THREE, III., and 3. The ideas thus arising, and similar ideas, are called numbers; hence,

A NUMBER is a unit, or a collection of units

IDEAS OF NUMBERS GENERALIZED.

§ 69. If we begin with the idea of the number one, and then add it to one, making two; and then add it to two, making three; and then to three, making four; and then to four, making five, and so on; it is plain that we shall form a series of numbers, each of which will be greater by one than that which precedes it. Now, one or a unit, is the base of this series of numbers, and each number may be expressed in three ways:

1st. By the words ONE, TWO, THREE, &c., of our common language;

2d. By the Roman characters; and,

3d. By figures.

§ 70. Since all numbers, whether integral or fractional, must come from, and hence be connected with, the unit one, it follows that there is but one purely elementary idea in the science of numbers. Hence, the idea of every number, regarded as made up of units (and all numbers except one must be so regarded,

when we analyze them), is necessarily complex. For, since the number arises from the addition of ones, the apprehension of it is incomplete until we understand how these additions were made; and therefore, a full idea of any number is necessarily complex.

§ 71. But if we regard a number as an entirety, that is, as an entire or whole thing, as an entire two, or three, or four, without pausing to analyze the units of which it is made up, it may then be regarded as a simple or incomplex idea; though, as we have seen, such idea may always be traced to that of the unit one, which forms the base of the number.

UNITY AND A UNIT DEFINED.

§ 72. When we name a number, as twenty feet, two things are necessary to its clear apprehension.

1st. A distinct apprehension of the *single thing* which forms the base of the number; and,

2d. A distinct apprehension of the *number of times* which that thing is taken.

Any number, or thing, regarded as an entirety, or whole, is called, **UNITY**.

A single thing which forms the base of a number, is called, a *unit*.

For example, one foot, one rod, or twenty rods, regarded as a standard of measure, is called. *unity*:

but one foot considered as one of the twenty equal parts of twenty feet, is called a UNIT, and is the base of the number twenty feet.

ALPHABET—WORDS—GRAMMAR.

§ 73. The term alphabet, in its most general sense, denotes a set of characters which form the elements of a written language.

When any one of these characters, or any combination of them, is used as the sign of a distinct notion or idea, it is called a word; and the naming of the characters of which the word is composed, is called, its spelling.

Grammar, as a science, treats of the established connection between words, as the signs of ideas.

ARITHMETICAL ALPHABET.

§ 74. The arithmetical alphabet consists of ten characters, called figures. They are,

Naught,	One,	Two,	Three,	Four,	Five,	Six,	Seven,	Eight,	Nine,
0	1	2	3	4	5	6	7	8	9

and each may be regarded as a word, since it stands for a distinct idea.

WORDS—SPELLING AND READING IN ADDITION.

§ 75. The idea of one, being elementary, the character 1 which represents it, is also elementary, and

hence, cannot be spelled by the other characters of the Arithmetical Alphabet. But the idea which is expressed by 2 comes from the addition of 1 and 1: hence, the word represented by the character 2, may be spelled by 1 and 1. Thus, 1 and 1 are 2, is the arithmetical spelling of the word two.

Three is spelled thus: 1 and 2 are 3; and also, 2 and 1 are 3.

Four is spelled, 1 and 3 are 4; 3 and 1 are 4; 2 and 2 are 4.

Five is spelled, 1 and 4 are 5; 4 and 1 are 5; 2 and 3 are 5; 3 and 2 are 5.

Six is spelled, 1 and 5 are 6; 5 and 1 are 6; 2 and 4 are 6; 4 and 2 are 6; 3 and 3 are 6.

§ 76. In a similar manner, any number in arithmetic may be spelled; and hence, we see that the process of spelling in addition consists simply, in naming any two elements which will make up the number. All the numbers in addition are therefore spelled with two syllables. The *reading* consists in naming only the word which expresses the final idea. Thus,

0	1	2	3	4	5	6	7	8	9
1	1	1	1	1	1	1	1	1	1
One	two	three	four	five	six	seven	eight	nine	ten.

We may now read the words which express the first hundred combinations.

READINGS.

1	2	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1	1	1	1
1	2	3	4	5	6	7	8	9	10
2	2	2	2	2	2	2	2	2	2
1	2	3	4	5	6	7	8	9	10
3	3	3	3	3	3	3	3	3	3
1	2	3	4	5	6	7	8	9	10
4	4	4	4	4	4	4	4	4	4
1	2	3	4	5	6	7	8	9	10
5	5	5	5	5	5	5	5	5	5
1	2	3	4	5	6	7	8	9	10
6	6	6	6	6	6	6	6	6	6
1	2	3	4	5	6	7	8	9	10
7	7	7	7	7	7	7	7	7	7
1	2	3	4	5	6	7	8	9	10
8	8	8	8	8	8	8	8	8	8
1	2	3	4	5	6	7	8	9	10
9	9	9	9	9	9	9	9	9	9
1	2	3	4	5	6	7	8	9	10
10	10	10	10	10	10	10	10	10	10

§ 77. In this example, beginning at the right hand, we say, 8, 17, 18, 26: setting down the 6 and carrying the 2, we say, 8, 13, 20, 22, 29: setting down the 9 and carrying the 2, we say, 9, 12, 18, 22, 30: and setting down the 30, we have the entire sum, 3096. All the examples in addition may be read in a similar manner.

878
421
679
354
764
3096

§ 78. The advantages of this method of reading, over spelling, are very great.

1st. The mind acquires ideas more readily through the eye than through either of the other senses. Hence, if the mind be taught to apprehend the result of a combination, by merely *seeing* its elements, the process of arriving at it is much shorter than when those elements are presented through the instrumentality of sound. Thus, to *see* 4 and 4, and *think* 8, is a very different thing from saying, four and four are eight.

2d. The mind operates with greater rapidity and certainty, the nearer it is brought to the ideas which it is to apprehend and combine. Therefore, all unnecessary words load it and impede its operations. Hence, to spell when we can read, is to fill the mind with words and sounds, instead of ideas.

3d. All the operations of arithmetic, beyond the elementary combinations, are performed on paper; and

if rapidly and accurately done, must be done through the eye and by reading. Hence the great importance of beginning early with a method which must be acquired before any considerable skill can be attained in the use of figures.

§ 79. It must not be supposed that the *reading* can be accomplished until the *spelling* has first been learned.

In our common language, we first learn the alphabet, then we pronounce each letter in a word, and finally, we pronounce the word. We should do the same in the arithmetical readings.

WORDS—SPELLING AND READING IN SUBTRACTION.

§ 80. The processes of spelling and reading which we have explained in the addition of numbers, may, with slight modifications, be applied in subtraction. Thus, if we are to subtract 2 from 5, we say, ordinarily, 2 from 5 leaves 3; or, 2 from 5, three remains. Now, the word, three, is suggested by the relation in which 2 and 5 stand to each other, and this word may be read at once. Hence, *the reading, in subtraction, is simply naming the word, which expresses the difference between the minuend and subtrahend.* Thus, we may read each word of the following one hundred combinations.

SPELLING AND READING IN MULTIPLICATION.

In a similar manner we read the entire multiplication table.

12 11 10 9 8 7 6 5 4 3 2 1
1

12	11	10	9	8	7	6	5	4	3	2	1 2
12	11	10	9	8	7	6	5	4	3	2	1 3
12	11	10	9	8	7	6	5	4	3	2	1 4
12	11	10	9	8	7	6	5	4	3	2	1 5
12	11	10	9	8	7	6	5	4	3	2	1 6
12	11	10	9	8	7	6	5	4	3	2	1 7
12	11	10	9	8	7	6	5	4	3	2	1 8
12	11	10	9	8	7	6	5	4	3	2	1 9
12	11	10	9	8	7	6	5	4	3	2	1 10
12	11	10	9	8	7	6	5	4	3	2	1 11
12	11	10	9	8	7	6	5	4	3	2	1 12

SPELLING AND READING IN DIVISION.

§ 83. In all the cases of short division, the quotient may be read immediately without naming the process by which it is obtained. Thus, in dividing the following numbers by 2, we merely read the words below.

2) 4	6	8	10	12	16	18	22
two	three	four	five	six	eight	nine	eleven.

In a similar manner, all the words, expressing the results in short division, may be read.

READINGS.

2) 2	4	6	8	10	12	14	16	18	20	22	24
3) 3	6	9	12	15	18	21	24	27	30	33	36
4) 4	8	12	16	20	24	28	32	36	40	44	48
5) 5	10	15	20	25	30	35	40	45	50	55	60
6) 6	12	18	24	30	36	42	48	54	60	66	72
7) 7	14	21	28	35	42	49	56	63	70	77	84
8) 8	16	24	32	40	48	56	64	72	80	88	96
9) 9	18	27	36	45	54	63	72	81	90	99	108

10) 10 20 30 40 50 60 70 80 90 100 110 120

11) 11 22 33 44 55 66 77 88 99 110 121 132

12) 12 24 36 48 60 72 84 96 108 120 132 144

UNITS INCREASING BY THE SCALE OF TENS.

§ 84. The idea of a particular number is necessarily complex; for, the mind naturally asks:

- 1st. What is the unit or base of the number? and,
- 2d. How many times is the unit or base taken?

§ 85. A figure indicates how many times a unit is taken. Each of the ten figures, however written, or however placed, always expresses as many units as its name imports, and no more; nor, does the *figure itself* at all indicate the kind of unit.

Still, every number expressed by one or more figures, has for its base either the abstract unit one, or a denominate unit. If a denominate unit, its value or kind is pointed out either by our common language, or, as we shall presently see, by the *place* where the figure is written.

The *number* of units which may be expressed by either of the ten figures, is indicated by the name of the figure. If the figure stands alone, and the

unit is not denominated, the base of the number is the abstract unit 1.

§ 86. If we write 0 on the right of 1, we }
 have } 10,
 which is read ONE ten. Here 1 still expresses ONE but it is ONE ten; that is, a unit ten times as great as the unit 1; and this is called a unit of the *second order*.

Again; if we write two 0's on the right of }
 1, we have } 100,
 which is read ONE hundred. Here again, 1 still expresses ONE, but it is ONE hundred; that is, a unit ten times as great as the unit ONE ten, and a hundred times as great as the unit 1.

§ 87. If three 1's are written by the side }
 of each other, thus } 111,
 the ideas, expressed in our common language, are these:

1st. *That the 1 on the right, will either express a single thing denominated, or the abstract unit one.*

2d. *That the 1 next at the left, expresses 1 ten, that is, a unit ten times as great as the first;*

3d. *That the 1 still further to the left, expresses 1 hundred; that is, a unit ten times as great as the second, and one hundred times as great as the first; and similarly if there are other places.*

in and of itself. The *number* of things, which any figure expresses, is determined by its name, as given in the arithmetical alphabet. The *kind* of thing, or unit of the figure, is fixed either by naming it, as in the case of a denominate number, or by the place which the figure occupies, when written by the side of or over other figures.

The phrase "local value of a figure," so long in use, is, therefore, without signification when applied to a figure: the term "local value," being applicable to the *unit of the place*, and not to the figure which occupies the place.

§ 89. A *SCALE* is a series of numbers expressing the law of relation between the different units of any number.

A *Uniform Scale* is one in which the law of relation between the units, at any step of the scale, is the same.

A *Varying Scale* is one in which the law of relation between the units, is different, at different steps of the scale.

The *Units of the Scale*, at any step, are denoted by the number of units of the lower denomination which makes one unit of the next higher.

§ 90. United States money affords an example of a series of denominate units, increasing according to the scale of tens: thus,

Eagle,	Dollar,	Dime,	Cent,	Mill,
1	1	1	1	1

may be read 11 thousand 1 hundred and 11 *mills*; or, 1111 *cents* and 1 mill; or, 111 dimes 1 cent and 1 mill; or, 11 dollars 1 dime 1 cent and 1 mill; or, 1 eagle 1 dollar 1 dime 1 cent and 1 mill. Thus, we may read the number with either of its units as a base, or we may name them all: thus, 1 eagle, 1 dollar, 1 dime, 1 cent, 1 mill. Generally; in United States Money, we read in the denominations of dollars, cents, and mills; and should say, 11 dollars 11 cents and 1 mill.

§ 91. Examples in reading figures:—

If we have the figures 89
we may read them by their smallest unit,
and say eighty-nine; or, we may say 8 tens and 9
units.

Again, the figures 567
may be read by the smallest unit; viz.
five hundred and sixty-seven; or we may say, 56
tens and 7 units; or, 5 hundreds 6 tens and 7
units.

Again, the number expressed by . . 74896
may be read, seventy-four thousand eight
hundred and ninety-six. Or, it may be read, 7489

tens and 6 units; or, 748 hundreds 9 tens and 6 units; or, 74 thousands 8 hundreds 9 tens and 6 units; or, 7 ten thousands 4 thousands 8 hundreds 9 tens and 6 units; and we may read in a similar way all other numbers.

Although we should teach all the correct readings of a number, we should not fail to remark that it is generally most convenient in practice to read by the lowest unit of a number. Thus, in the numeration table, we read each period by the lowest unit of that period. For example, in the number

874,967,847,047,

we read 874 *billions* 967 *millions* 847 *thousands* and 47.

UNITS INCREASING ACCORDING TO VARYING SCALES.

§ 92. If we write the well-known signs of the English money, and place 1 under each denomination, we shall have

£	s.	d.	f.
1	1	1	1

Now, the signs £ s. d. and f. fix the value of the unit 1 in each denomination; and they also determine the relations which subsist between the differ-

ent units. For example, this simple language expresses these ideas :

1st. That the unit of the right-hand place is 1 farthing—of the place next to the left, 1 penny—of the next place, 1 shilling—of the next place, 1 pound ; and

2. That 4 units of the lowest denomination make one unit of the next higher ; 12 of the second, one of the third ; and 20 of the third, one of the fourth.

§ 93. If we take the denominate numbers of the Avoirdupois weight, we have

<i>Ton</i>	<i>cwt.</i>	<i>qr.</i>	<i>lb.</i>	<i>oz.</i>	<i>dr.</i>
1	1	1	1	1	1 ;

in which the units increase in the following manner : viz. the second unit, counting from the right, is sixteen times as great as the first ; the third, sixteen times as great as the second ; the fourth, twenty-five times as great as the third ; the fifth, four times as great as the fourth : and the sixth, twenty times as great as the fifth. The scale, therefore, for this class of denominate numbers varies according to the above laws.

If we take any other class of denominate numbers, as the Troy weight, or any of the systems of measures, we shall have different scales for the formation of the different units. But in all the formations, we

recognize the application of the same general principles.

There are, therefore, two general methods of forming the different systems of integral numbers from the unit one. The first consists in preserving a constant law of relation between the different unities; viz. that their values shall change according to the scale of tens. This gives the system of common numbers.

The second method consists in the application of known, though varying laws of change in the unities. These changes in the unities produce the entire system of denominate numbers, each class of which has its appropriate scale; and the changes among the units, of the same class, are indicated at the different steps of the scale.

INTEGRAL UNITS OF ARITHMETIC.

§ 94. There are eight classes of units in Arithmetic:

- 1st. Abstract Units.
- 2d. Units of Currency.
- 3d. Units of Weight.
- 4th. Units of Time.
- 5th. Units of Length.
- 6th. Units of Surface.
- 7th. Units of Volume.
- 8th. Units of Angular Measure.

First among the Units of arithmetic stands the ab-

stract unit 1. This is the base of all abstract numbers, and becomes the base, also, of all denominate numbers, by merely naming, in succession, the particular things to which it is applied.

It is also the base of all fractions. Merely as the unit 1, it is a whole which may be divided according to any law, forming every variety of fraction; and if we apply it to a particular thing, the fraction becomes denominate, and we have expressions for all conceivable parts of that thing.

§ 95. It has been remarked that we can form no distinct apprehension of the number, until we have a clear notion of its unit, and the number of times the unit is taken. *The unit is the great base.* The utmost care, therefore, should be taken to impress on the minds of learners, a clear and distinct idea of the actual value of the unit of every number with which they have to do. If it be a number expressing currency, one or more of the coins should be exhibited, and the value dwelt upon; after which, distinct notions of the other units can be acquired by comparison.

If the number be one of weight, some unit should be exhibited, as one pound, or one ounce, and an idea of its weight acquired by actually lifting it. This is the only way in which we can learn the true signification of the terms.

If the number be one of measure, either linear, superficial, of volumes or angles, its unit should also be exhibited, and *the signification of the term expressing it, learned in the only way in which it can be learned, through the senses, and by the aid of a sensible object.*

ABSTRACT UNITS.

§ 96. Abstract units, are those which denote single things, in which the base is the abstract number one. They are mere expressions of number, and may be applied to all quantities by assigning the proper denominate unit.

ORDERS OF ABSTRACT UNITS.

§ 97. The abstract units, are, one, one ten, one hundred, one thousand, one ten thousand, &c., &c. They are called units of different orders. Thus, one is a unit of the *first* order; ten of the *second*; one hundred of the *third*; one thousand of the fourth, &c.

UNITS OF CURRENCY.

§ 98. The Units of United States currency, are 1 mill, 1 cent, 1 dime, 1 dollar, and 1 eagle. The law of change, in passing from one unit to another, is according to the scale of tens. Hence, this system of

numbers may be treated, in all respects, as abstract numbers; and indeed they are such, with the single exception that their units have different names.

They are generally read in the units of dollars, cents, and mills—a period being placed after the figure denoting dollars. Thus,

\$ 864.849

is read, eight hundred and sixty-four dollars, eighty-four cents and nine mills; and if there were a figure after the 9, it would be read in decimals of the mill. The number may, however, be read in any other unit; as, 864849 mills; or, 86484 cents and 9 mills; or, 8648 dimes, 4 cents, and 9 mills; or, 86 eagles, 4 dollars, 84 cents, and 9 mills; and there are yet several other readings.

ENGLISH CURRENCY.

§ 99. The units of English, or Sterling Money, are 1 farthing, 1 penny, 1 shilling, and 1 pound.

The scale of this class of numbers is a varying scale. Its steps, in passing from the unit of the lowest denomination to the highest, are four, twelve, and twenty. For, four farthings make one penny, twelve pence one shilling, and twenty shillings one pound.

-FRENCH CURRENCY.

§ 100. The units of the French Currency are the Franc, which forms the base, and the Napoleon, equal to twenty francs.

UNITS OF WEIGHT.

§ 101. There are three kinds of Weight in general use: Avoirdupois Weight, Troy Weight, and Apothecaries' Weight.

AVOIRDUPOIS WEIGHT.

§ 102. The units of the Avoirdupois Weight are, 1 dram, 1 ounce, 1 pound, 1 quarter, 1 hundred-weight, and 1 ton.

The scale of this class of numbers is a varying scale. Its steps, in passing from the unit of the lowest denomination to the highest, are sixteen, sixteen, twenty-five, four, and twenty. For, sixteen drams make one ounce, sixteen ounces one pound, twenty-five pounds one quarter, four quarters one hundred, and twenty hundreds one ton.

TROY WEIGHT.

§ 103. The units of the Troy Weight are, 1 grain, 1 pennyweight, 1 ounce, and 1 pound.

The scale is a varying scale; and its steps, in passing from the unit of the lowest denomination to the highest, are twenty-four, twenty, and twelve.

APOTHECARIES' WEIGHT.

§ 104. The units of this weight are, 1 grain, 1 scruple, 1 dram, 1 ounce, and 1 pound.

The scale is a varying scale. Its steps, in passing from the unit of the lowest denomination to the highest, are twenty, three, eight, and twelve.

TIME.

§ 105. The units of Time are, 1 second, 1 minute, 1 hour, 1 day, 1 week, 1 month, 1 year, and 1 century. The steps of the scale, in passing from units of the lowest denomination to the highest, are sixty, sixty, twenty-four, seven, four, twelve, and one hundred.

UNITS OF LENGTH.

§ 106. The unit of Length is used for measuring lines, either straight or curved. It is a straight line of a given length, and is called the standard or base, of the measurement.

The units of length, generally used as standards, are 1 inch, 1 foot, 1 yard, 1 rod, 1 furlong, and 1 mile.

The number of times which the unit, used as a standard, is taken, considered in connection with its value, gives the idea of the length of the line measured.

UNITS OF SURFACE.

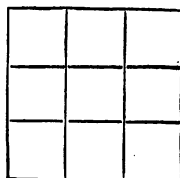
§ 107. Units of surface are used for the measurement of the area or contents of whatever has the two dimensions of length and breadth. The unit of surface is a square described on the unit of length as a side. Thus, if the unit of length be 1 foot, the corresponding unit of surface will be 1 square foot; that is, a square constructed on 1 foot of length, as a side.

1 square foot.



If the linear unit be 1 yard, the corresponding unit of surface will be 1 square yard. It will be seen from the figure, that, although the linear yard contains the linear foot but three times, the square yard contains the square foot nine times. The square rod or square mile may also be used as the unit of surface.

1 yard.



The number of times which a surface contains its unit of measure, is its area or contents; and this number, taken in connection with the value of the unit, gives the idea of its extent.

Besides the units of surface already considered, there is another kind, called,

DUODECIMAL UNITS.

§ 108. The duodecimal units are generally used in board measure, though they may be used in all superficial measurements, and also in volumes.

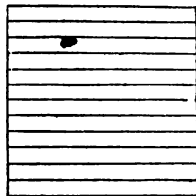
The square foot is the base of this class of units, and the others are deduced from it, by a descending scale of twelve.

§ 109. It is proved in Geometry, that if the number of linear units in the base of a rectangle be multiplied by the number of linear units in its height, the numerical value of the product will be equal to the number of superficial units in the figure.

Knowing this fact, we often express it by saying, that "feet multiplied by feet give square feet," and "yards multiplied by yards give square yards." But as feet cannot be taken *feet times*, nor yards *yard times*, this language, rightly understood, is but a concise form of expression for the principle stated above.

With this understanding of the language, we say, that 1 foot in length multiplied by 1 foot in height, gives a square foot; and 4 feet in length multiplied by 3 feet in height, gives 12 square feet.

§ 110. If now, 1 foot in length be multiplied by 1 inch = $\frac{1}{12}$ of a foot in height, the product will be one-twelfth of a square foot; that is, *one-twelfth of the first unit*: if it be multiplied by 3 inches, the pro-



duct will be three-twelfths of a square foot; and similarly, for a multiplier of any number of inches.

If, now, we multiply 1 inch by 1 inch, the product may be represented by 1 square inch: that is, *by one-twelfth of the last unit*. Hence, *the units of this measure decrease according to the scale of 12*. The units are,

1st. Square feet—arising from multiplying feet by feet:

2d. Twelfths of square feet—arising from multiplying the denomination of feet by the denomination of inches:

3d. Twelfths of twelfths—arising from multiplying inches by inches.

The same remarks apply to the smaller divisions of the foot, according to the scale of twelve.

The difficulty of computing, in this measure, arises from the changes in the units.

UNITS OF VOLUME.

§ 111. VOLUME has been defined in Geometry to be a limited portion of space. It has already been stated,

that if length be multiplied by breadth, the product may be represented by units of surface. It is also proved, in Geometry, that if the length, breadth, and height of any regular figure, of a square form, be multiplied together, the product may be represented by units of volume whose number is equal to this product. Each unit of volume is a cube constructed on the linear unit as an edge. Thus, if the linear unit be 1 foot, the unit of volume will be 1 cubic foot; that is, a cube constructed on 1 foot as an edge; and if it be 1 yard, the unit will be 1 cubic yard.

The three units, viz. the unit of length, the unit of surface, and the unit of volume, are essentially different in kind. The first is a line of a known length; the second, a square of a known side; and the third, a volume, called a cube, of a known base and height. These are the units used in all kinds of measurements of solids, excepting only the duodecimal system, which has already been explained.

§ 112. A volume filled with matter, is called a *solid* or *body*; and for this, the unit of measure is the cube, except, in the case of wood, where it is a cord of 128 cubic feet.

But a given volume filled with a liquid, takes the name of a gallon; and a larger volume filled with grain, takes the name of a bushel. Hence, the Liquid and Dry measures, are measures of volume, and differ

only from the measurement of solidity, in the form and value of the unit of measure.

LIQUID MEASURE.

§ 113. The units of Liquid Measure are, 1 gill, 1 pint, 1 quart, 1 gallon, 1 barrel, 1 hogshead, 1 pipe, 1 tun. The scale is a varying scale. Its steps, in passing from the unit of the lowest denomination, are, four, two, four, thirty-one and a half, sixty-three, two, and two.

DRY MEASURE.

§ 114. The units of this measure are, 1 pint, 1 quart, 1 peck, 1 bushel, and 1 chaldron. The steps of the scale, in passing from units of the lowest denomination, are, two, eight, four, and thirty-six.

ANGULAR MEASURE.

§ 115. The units of this measure are, 1 second, 1 minute, 1 degree, 1 sign, 1 circle. The steps of the scale, in passing from units of the lowest denomination to those of the higher, are sixty, sixty, thirty, and twelve.

ADVANTAGES OF THE SYSTEM OF UNITIES.

§ 116. It may well be asked, if the method here adopted, of presenting the elementary principles of arithmetic, has any advantages over those now in general use. It is supposed to possess the following:

1st. The system of unities teaches an exact analysis of all numbers, and unfolds to the mind the different ways in which they are formed from the unit one, as a base.

2d. Such an analysis enables the mind to form a definite and distinct idea of every number, by pointing out the relation between it and the unit from which it was derived.

3d. By presenting constantly to the mind the idea of the unit one, as the base of all numbers, the mind is insensibly led to compare this unit with all the numbers which flow from it, and then it can the more easily compare those numbers with each other.

4th. It affords a more satisfactory analysis and a better understanding of the four ground-rules, and indeed of all the operations of arithmetic, than any other method of presenting the subject.

FOUR GROUND-RULES.

§ 117. Let us take the two following examples in Addition, the one in abstract and the other in

denominate numbers, and then analyze the process of finding the sum in each.

SIMPLE NUMBERS.

874198

36984

3641

914823

DENOMINATE NUMBERS.

cut. gr. lb. oz. dr.

3 3 24 15 14

6 3 23 14 8

10 3 23 14 6

In both examples we begin by adding the units of the lowest denomination, and then, we *divide* their sum by so many as make one of the denomination next higher. We then set down the remainder, and add the quotient to the units of that denomination. Having done this, we apply a similar process to all the other denominations—the principle being precisely the same in both examples. We see, in these examples, an illustration of a general principle of addition, viz. *that units of the same kind are always added together.*

§ 118. Let us take two similar examples in Subtraction.

SIMPLE NUMBERS.

8403

3298

5105

DENOMINATE NUMBERS.

£ s. d. far.

6 9 7 2

3 10 8 4

2 18 10 2

In both examples we begin with the units of the lowest denomination, and as the number in the subtrahend is greater than in the place directly above, we suppose so many to be added in the minuend as make one unit of the next higher denomination. We then make the subtraction, and add 1 to the units of the subtrahend next higher, and proceed in a similar manner, through all the denominations. It is plain that the principle employed is the same in both examples. Also, that units of any denomination in the subtrahend are taken from those of the same denomination in the minuend.

§ 119. Let us now take similar examples in Multiplication.

SIMPLE NUMBERS.

$$\begin{array}{r} 87464 \\ 5 \\ \hline 437320 \\ \hline \end{array}$$

DENOMINATE NUMBERS.

$$\begin{array}{r} \text{lb} \quad \text{3} \quad \text{3} \quad \text{3} \quad \text{gr.} \\ 9 \quad 7 \quad 6 \quad 2 \quad 15 \\ \hline 48 \quad 3 \quad 2 \quad 1 \quad 15 \\ \hline \end{array}$$

In these examples we see, that we multiply, in succession, each order of units in the multiplicand by the multiplier, and that we carry from one product to another, one for every so many as make one unit of the next higher denomination. The

principle of the process is therefore the same in both examples.

LOGICAL FORM FOR ADDITION.

§ 120. *Def.* A number which contains as many units as all the numbers added, is called their *sum*.

With this definition, let it be required to *prove* that 8 is the sum of 3 and 5.

The Syllogistic form is:

A number which contains as many units as all the numbers added, is called their sum. (Major Premise.)

Eight contains as many units as there are in 3 and 5; (because 3 counted on to 5 make 8). (Minor Premise.)

Therefore, 8 is the sum of 3 and 5. This proof, logically, is perfect, because it brings the result of the operation performed on 3 and 5, *under* the term, Sum.

LOGICAL FORM FOR SUBTRACTION.

§ 121. *Def.* A number which added to the less of two numbers will give the greater, is called their *Difference*.

What is the difference between 7 and 10?

The syllogistic form is:

A number which added to the less of two numbers will give the greater, is called their difference; 3 added to 7 gives 10;

Therefore, 3 is the difference between 7 and 10.

Here, again, the logical form merely brings the result of the operation under the definition.

§ 122. In Multiplication, if we define the operation to be, the process of taking the multiplicand as many times as there are units in the multiplier, we prove the operation by showing that the result fulfils this condition.

§ 123. So, in Division, if we define the quotient to be such a number as multiplied by the divisor will produce the dividend; we prove the operation to be correct, when we show that the number found is such a multiplier.

Every proof in the entire range of mathematical science, consists in bringing the thing to be proved, under a definition, an axiom, or a proposition previously established.

METRIC SYSTEM.

§ 124. Every system of Weights and Measures *must* have an *invariable unit* for its base—and every other unit of the entire system should be derived from it, according to a fixed law.

The French Government, in order to obtain an invariable unit, measured a degree of the arc of a

meridian on the Earth's surface; and from this computed the length of the meridional arc from the equator to the pole. This length they divided into ten million equal parts, and then took one of these parts for the unit of length, and called it a **METER**. The length of this meter is equal to 1 yard, 3 inches and 37 hundredths of an inch, very nearly. Thus they obtained the length of the unit which is the base of the Metric System of Weights and Measures.

The next step was to fix the *law* by which the other units of the system should be obtained from the base. As the scale of tens is the simplest law by which we can pass from one unit to another, that scale was adopted, and the larger units are formed by multiplying the base continually by 10, and the smaller, by dividing it continually by 10.

NAMING.

§ 125. The names, in the ascending scale, are formed by prefixing to the base, Meter, the words Deca (ten), Hecto (one hundred), Kilo (one thousand), Myria (ten thousand), from the Greek numerals; and in the descending scale, by prefixing Deci (tenths), centi (hundredth), Milli (thousandth), from the Latin numerals. Hence, the name of a unit indicates whether it is *greater* or *less* than the standard, and also, how many times.

MEASUREMENT OF LENGTH

Hence, for the measurement of length, we have

<i>Ascending Scale.</i>				<i>Unit.</i>	<i>Descending Scale.</i>		
Myriameter	Kilometer	Hectometer	Decameter		Decimeter	Centimeter	Millimeter

in which the increase and decrease, from the base, takes place according to the scale of tens.

SQUARE MEASURE.

§ 126. The unit of measure for surfaces is a square described on a line 10 meters in length, and is called an **ARE**. Hence, the unit of surface is equal to 100 square meters. It is also equal to 4 perches, nearly. There are but two other units used in the French system—the **CENTARE**, which is one hundredth of the **ARE**, and the **HECTARE**, which is one hundred times the **ARE**.

MEASURES OF VOLUME.

§ 127. The unit for the measurement of volumes is the cube constructed on the decimeter, as an edge. It is called a **LETER**; and is equal to 61 cubic inches, very nearly. All the other denominations are derived from

the base, in the same manner as in the measurement of length. Thus, we have

<i>Ascending Scale.</i>			<i>Unit.</i>	<i>Descending Scale.</i>		
Kiloleter	Hectoleter	Decaleter		Decileter	Centileter	Millileter

DRY MEASURE.

§ 128. The **LETER** is also the unit for dry measure. It is a little less than 1 quart of the Winchester bushel.

LIQUID MEASURE.

§ 129. The unit of this measure is also the **LETER**. It contains a little more than 3 pints of the wine gallon.

WEIGHTS.

§ 130. The unit of weight is the weight of a cubic centimeter of pure rain-water, weighed in vacuum. Hence, it is equal to the one-millionth part of the weight of a cubic meter of pure rain-water. It is called a **GRAM**, and is equal to 15.423 grains Troy, which is equal to .03527 ounces Avoirdupois, very nearly. All the other denominations are formed from the base as before. Hence, we have:

<i>Ascending Scale.</i>					<i>Unit.</i>	<i>Descending Scale.</i>		
Millier	Quintal	Myriagram	Kilogram	Hectogram		Decigram	Centigram	Milligram

NATURE OF THE METRIC SYSTEM.

§ 131. The Metric System is based on the **METER**. From the Meter, three other units are derived; and the four constitute the four primary bases of this system of weights and measures. They are

METER = 39.37 inches; unit of length.

ARE = a square on 10 meters; unit of surface.

LETER = a cube whose edge is a decimeter; unit of volume.

GRAM = weight of a cube of rain-water, each edge of which is a centimeter; unit of weight.

From these four units all the others are derived, according to the decimal scale, as already explained. All, therefore, which is necessary to a full and complete apprehension of the Metric System, is to comprehend the meaning of the four names, Meter, Are, Leter, and Gram—of the Latin prefixes, Deci, Centi, Milli, of the descending scale—and of the Greek prefixes, Deca, Hecto, Kilo, Myria, of the ascending scale;

and then remember, that every change from one unit to another, is according to the scale of tens. This system, therefore, requires only the knowledge of eleven words, and of only one, and that, the simplest, law of number, viz., the change from one unit to another according to the scale of tens. It would seem impossible, that so simple a system of computation and record, forming a common language for the whole family of man, and reaching every operation of trade and commerce, should not, at an early day, become universal.

FRACTIONAL UNITS.

FRACTIONAL UNITS.—SCALE OF TENS.

§ 132. If the unit 1 be divided into ten equal parts, each part is called *one tenth*. If one of these tenths be divided into ten equal parts, each part is called *one hundredth*. If one of the hundredths be divided into ten equal parts, each part is called *one thousandth*; and corresponding names are given to similar parts, how far soever the divisions may be carried.

Now, although the tenths which arise from dividing the unit 1, are but equal parts of 1, they are,

nevertheless, WHOLE tenths, and in this light may be regarded as *units*.

To avoid confusion, in the use of terms, we shall call every equal part of 1, a *fractional unit*. Hence, tenths, hundredths, thousandths, tenths of thousandths, &c., are *fractional units*, each having a *fixed relation to the unit 1*, from which it was derived.

§ 133. Adopting a similar language to that used in integral numbers, we call the tenths, fractional units of the *first order*; the hundredths, fractional units of the *second order*; the thousandths, fractional units of the *third order*; and so on for the subsequent divisions.

Is there any arithmetical language by which these fractional units may be expressed? The decimal point, which is merely a dot, or period, indicates the division of the unit 1, according to the scale of tens. By the arithmetical language, the unit of the place next the point, on the right, is 1 tenth; that of the second place, 1 hundredth; that of the third, 1 thousandth; that of the fourth, 1 ten-thousandth; and so on for places still to the right.

The scale for decimals, therefore, is

.00000000, &c.;

in which the unit of each place is known as soon as we have learned the signification of the language.

If, therefore, we wish to express any of the parts into which the unit 1 may be divided, according to the scale of tens, we have simply to select from the alphabet, the figure that will express the *number* of parts, and then write it in the place corresponding to the *order of the unit*. Thus, to express four tenths, three thousandths, eight ten-thousandths, and six millionths, we write

$$.403806;$$

and similarly, for any decimal which can be named.

§ 134. It should be observed that while the units of place *decrease*, according to the scale of tens, from left to right, they *increase* according to the same scale, from right to left. *This is the same law of increase as that which connects the units of place in abstract numbers.* Hence, abstract numbers and decimals being formed according to the same law, may be written by the side of each other, and treated as a single number, by merely preserving the separating or decimal point. Thus, 8974 and .67046 may be written

$$8974.67046;$$

since ten units, in the place of tenths, make the unit one, in the place next to the left.

FRACTIONAL UNITS IN GENERAL.

§ 135. If the unit 1 be divided into two equal parts, each part is called a half. If it be divided into three equal parts, each part is called a third: if it be divided into four equal parts, each part is called a fourth: if into five equal parts, each part is called a fifth; and if into any other number of equal parts, a name is given corresponding to the number of parts.

Now, although these halves, thirds, fourths, fifths, &c., are each but parts of the unit 1, they are, nevertheless, in *themselves*, whole things. That is, a half, is a whole half; a third, a whole third; a fourth, a whole fourth; and the same for any other equal part of 1. In this sense, therefore, they are *units*, and we call them fractional units. Each is an exact part of the unit 1, and has a fixed relation to it.

§ 136. Is there any arithmetical language by which these fractional units can be expressed?

The bar written at the right, is the sign which denotes the division of the unit 1 into any number of equal parts.

If we wish to express the number of equal parts into which it is divided, as 9, for example, we simply write the 9 under the bar, and then the phrase means, that some thing

—

 $\overline{9}$

regarded as a whole, has been divided into 9 equal parts.

If, now, we wish to express any number of these fractional units, as 7, for example, we place the 7 above the line, and read, $\frac{7}{9}$ seven ninths.

§ 137. It has been observed, that two things are necessary to the clear apprehension of an integral number.

1st. A distinct apprehension of the *unit* which forms the base of the number; and,

2d. A distinct apprehension of the number of times which that unit is taken.

Three things are necessary to the distinct apprehension of the value of any fraction, either decimal or vulgar.

1st. We must know the unit, or whole thing, from which the fraction was derived;

2d. We must know into how many equal parts that unit is divided; and,

3d. We must know how many such parts are taken in the expression.

The unit from which the fraction is derived, is called the *unit of the fraction*; and one of the equal parts is called, the *fractional unit*.

For example, to apprehend the value of the fraction $\frac{3}{4}$ of a pound avoirdupois, or $\frac{3}{4}lb.$; we must know,

1st. What is meant by a pound;

2d. That it has been divided into seven equal parts; and,

3d. That three of those parts are taken.

In the above fraction, 1 pound is the unit of the fraction; one-seventh of a pound, is the fractional unit; and 3, denotes that three fractional units are taken.

If the unit of a fraction is not named, it is taken to be the abstract unit 1.

ADVANTAGES OF FRACTIONAL UNITS.

§ 138. By considering every equal part of a unit as an entire thing, having a certain relation to the unit 1, the mind is led to analyze a fraction, and thus to apprehend its precise signification.

Under this searching analysis, the mind at once seizes on the *unit of the fraction* as the principal base. It then looks at the value of each part. It then inquires how many such parts are taken.

It having been shown that equal integral units can alone be added, it is readily seen that the same principle is equally applicable to fractional units; and then the inquiry is made: What is necessary in order to make such units equal?

It is seen at once, that two things are necessary:

1st. That they be parts of the *same unit*; and,

2d. That they be *like parts*; in other words, they must be of the same denomination, and have a common denominator.

In regard to Decimal Fractions, all that is necessary, is to observe that units of the same value are added to each other, and when the figures expressing them are written down, they should always be placed in the same column.

§ 139. The great difficulty in the management of fractions, consists in comparing them with each other, instead of constantly comparing them with the unit from which they are derived. By considering them as entire things, having a fixed relation to the unit which is their base, they can be compared as readily as integral numbers; for, the mind is never at a loss when it apprehends the unit, the parts into which it is divided, and the number of parts which are taken. The only reasons why we apprehend and handle integral numbers more readily than fractions, are,

1st. Because the unit forming the base is always kept in view; and,

2d. Because, in integral numbers, we have been taught to trace constantly the connection between the unit and the numbers which come from it; while in the methods of treating fractions, these important considerations have been neglected.

PROPORTION AND RATIO.

§ 140. Proportion expresses the relation which one number bears to another, with respect to its being greater or less.

Two numbers may be compared, the one with the other, in two ways:

1st. With respect to their difference, called Arithmetical Proportion; and,

2d. With respect to their quotient, called Geometrical Proportion.

Thus, if we compare the numbers 1 and 8, by their difference, we find that the second exceeds the first by 7: hence, their difference 7, *is the measure of their arithmetical relation*, and is called their *arithmetical proportion*.

If we compare the same numbers by their quotient, we find that the second contains the first 8 times: hence, 8 is the *measure of their geometrical relation*, and is called their *ratio*.

§ 141. The two numbers which are thus compared, are called *terms*. The first is called the *antecedent*, and the second the *consequent*.

In comparing numbers with respect to their difference, the question is, *how much* is one greater or less than the other? Their difference affords the true answer, and is the measure of their proportion.

In comparing numbers with respect to their quotient, the question is, *how many times* is one greater or less than the other? Their quotient or ratio, is the true answer, and is the measure of their proportion. Ten, for example, is nine greater than one, if we compare the numbers one and ten by their difference. But if we compare them by their quotient, ten is said to be ten times as great as one—the language “ten times” having reference to the quotient, which is always taken as the measure of the relative value of the two numbers so compared. Thus, when we say, that, the units of our common system of numbers increase in a tenfold ratio, we mean that they so increase that each succeeding unit shall contain the preceding one ten times. This is a convenient language to express a particular relation of two numbers, and is perfectly correct, when used in conformity to an accurate definition.

§ 142. The first term is called the *standard*; and is always the unit of measure which measures the other.

§ 143. It may be proper here to observe, that while the term “geometrical proportion” is used to express the relation of two numbers, compared by their ratio, the term “A geometrical proportion,” is applied to four numbers, in which the ratio of the first to the



second is the same as that of the third to the fourth.

Thus,

$$2 : 4 :: 6 : 12,$$

is a geometrical proportion, of which the ratio is 2.

§ 144. One important remark on the subject of proportion is yet to be made. It is this:

Any two numbers which are compared together, either by their difference or quotient, must be of the same kind: that is, they must either have the same unit, as a base, or be susceptible of reduction to the same unit.

For example, we can compare 2 pounds with 6 pounds: their difference is 4 pounds, and their ratio is the abstract number 3. We can also compare 2 feet with 8 yards: for, although the unit 1 foot is different from the unit 1 yard, still 8 yards are equal to 24 feet. Hence, the difference of the numbers is 22 feet, and their ratio the abstract number 12.

On the other hand, we cannot compare 2 dollars with 2 yards of cloth, for they are quantities of different kinds, not being susceptible of reduction to a common unit.

Simple or abstract numbers may always be compared, since they have a common unit 1.

SECTION IV.

GEOMETRY.

§ 145. GEOMETRY treats of space, and compares portions of space with each other, for the purpose of pointing out their properties and mutual relations. The science consists in the development of all the laws relating to space, and is made up of the processes and rules, by means of which portions of space can be best compared with each other. The truths of Geometry are a series of dependent propositions, and may be divided into three classes:

1st. Truths implied in the definitions, viz. that things do exist, or may exist, corresponding to the words defined. For example, when we say, "A quadrilateral is a rectilinear figure having four sides," we imply the existence of such a figure.

2d. Self-evident, or intuitive truths, embodied in the axioms; and,

3d. Truths inferred or proved, from the definitions and axioms, called Demonstrative Truths. We say

that a truth or proposition is proved or demonstrated, when, by a course of reasoning, it is shown to be included under some other truth or proposition, previously known, and from which it is said to be derived; hence,

A DEMONSTRATION is a series of logical arguments, brought to a conclusion, in which the major premises are definitions, axioms, or propositions already established.

§ 146. Before we can understand the proofs or demonstrations of Geometry, we must understand what that is to which demonstration is applicable: hence, the first thing necessary is to form a clear conception of space, the subject of all geometrical reasoning.

The next step is to give names to particular forms or limited portions of space, and to define these names accurately. The definitions of these names are the definitions of Geometry, and the portions of space corresponding to them are called Figures, or Geometrical Magnitudes; of which there are four general classes:

- 1st. Lines;
- 2d. Surfaces;
- 3d. Volumes; and.
- 4th. Angles.

§ 147. Lines embrace only one dimension of space, viz. length, without breadth or thickness. The extremities, or limits of a line, are called points.

There are two general classes of lines—straight lines and curved lines. A straight line is one which lies in the same direction between any two of its points; and a curved line is one which constantly changes its direction at every point. There is but one kind of straight line, and that is fully characterized by the definition. From the definition we may infer the following axiom: “A straight line is the shortest distance between two points.” There are many kinds of curves, of which the circumference of the circle is the simplest and the most easily described.

§ 148. Surfaces embrace two dimensions of space, viz. length and breadth, but not thickness. Surfaces, like lines, are also divided into two general classes, viz. plane surfaces and curved surfaces.

A plane surface is that with which a straight line, any how placed, and having two points common with the surface, will coincide throughout its entire extent. Such a surface is perfectly even, and is commonly designated by the term “A plane.” A large class of the figures of Geometry are but portions of a plane, and all such are called plane figures.

§ 149. A portion of a plane, bounded by three straight lines, is called a triangle, and this is the simplest of the plane figures. There are several kinds of triangles, differing from each other, however, only in the relative values of their sides and angles. For example: when the sides are all equal to each other, the triangle is called equilateral; when two of the sides are equal, it is called isosceles; and scalene, when the three sides are all unequal. If one of the angles is a right angle, the triangle is called a right-angled triangle.

§ 150. The next simplest class of plane figures comprises all those which are bounded by four straight lines, and are called quadrilaterals. There are several varieties of this class:

1st. The mere quadrilateral, which has no mark, except that of having four sides called a trapezium;

2d. The trapezoid, which has two sides parallel and two not parallel;

3d. The parallelogram, which has its opposite sides parallel and its angles oblique;

4th. The rectangle, which has all its angles right angles and its opposite sides parallel; and,

5th. The square, which has its four sides equal to each other, each to each, and its four angles right angles.

§ 151. Plane figures, bounded by straight lines, having a number of sides greater than four, take names corresponding to the number of their sides, viz. Pentagons, Hexagons, Heptagons, &c., and the common term, Polygon, is applicable to every plane figure bounded by straight lines.

§ 152. A portion of a plane bounded by a curved line, all the points of which are equally distant from a certain point within called the centre, is called a circle, and the bounding line is called the circumference. This is the only curve usually treated of in Elementary Geometry.

§ 153. A curved surface, like a plane, embraces the two dimensions of length and breadth. It is not even, like the plane, throughout its whole extent, and therefore a straight line may have two points in common, and yet not coincide with it. The surface of the cone, of the sphere, and cylinder, are the curved surfaces treated of in Elementary Geometry.

§ 154. A volume is a portion of space, combining the three dimensions of length, breadth, and thickness. Volumes are divided into three classes:

- 1st. Those bounded by planes;
 - 2d. Those bounded by plane and curved surfaces;
- and,

3d. Those bounded only by curved surfaces.

The first class embraces the pyramid and prism with their several varieties; the second class embraces the cylinder and cone; and the third class the sphere, together with others not generally treated of in 'Elementary Geometry.'

§ 155. An angle, in Elementary Geometry, is the amount of inclination of two lines, or of two planes. Angles are generally compared with the right angle, which is their natural unit.

§ 156. We have now named all the geometrical magnitudes treated of in Elementary Geometry. They are merely limited portions of space, and do not, *necessarily*, involve the idea of matter. A sphere, for example, fulfils all the conditions imposed by its definition, without any reference to what may be within the space enclosed by its surface. That space may be occupied by lead, iron, or air, or may be a vacuum, without at all changing the nature of the sphere, as a geometrical magnitude.

It should be observed that the boundary or limit of a geometrical magnitude, is another geometrical magnitude, having one dimension less. For example: the boundary or limit of a volume, which has three dimensions, is always a surface which has but two: the limits or boundaries of all surfaces are lines,

straight or curved; and the extremities or limits of lines are points.

§ 157. We have now named and shown the nature of the things which are the subjects of Elementary Geometry. They are called, the Geometrical Magnitudes. The science of Geometry is a collection of those connected processes by which we determine the measures, properties, and relations of these magnitudes.

COMPARISON OF FIGURES WITH UNITS OF MEASURE.

§ 158. We have seen that the term *measure*, implies a comparison of the thing measured with some known thing of the same kind, regarded as a standard; and that such standard is called the *unit* of measure. The unit of measure for lines must, therefore, be a line of a known length; a foot, a yard, a rod, a mile, or any other known unit.

The unit of measure for surfaces, is a square constructed on the linear unit as a side: that is, a square foot, a square yard, a square rod, a square mile; that is, *a square described on any known unit of length*.

The unit of measure, for volumes, is a volume, and therefore has three dimensions. It is a cube constructed on a linear unit as an edge, or on the superficial unit as a base. It is, therefore, a cubic foot, a

cubic yard, a cubic rod, &c. The unit of measure for angles is the right angle. With this standard all angles are compared. Hence, there are four units of measure, each differing in kind from the other two, viz. a known length for the measurement of lines; a known square for the measurement of surfaces; a known volume for the measurement of volumes; and a known angle for the measurement of angles. The measure or contents of any magnitude, belonging to either class, is ascertained by finding how many times that magnitude contains its unit of measure.

§ 159. We have dwelt with much detail on the unit of measure, because it furnishes the only base of estimating quantity. The conception of number and space merely opens to the intellectual vision an unmeasured field of investigation and thought, as the ascent to the summit of a mountain presents to the eye a wide and unsurveyed horizon. To ascertain the height of the point of view, the diameter of the surrounding circular area and the distance to any point which may be seen, some standard or unit must be known, and its value distinctly apprehended. So, also, number and space, which at first fill the mind with vague and indefinite conceptions, are to be finally measured by units of ascertained value.

§ 160. It is found, on careful analysis, that every

number may be referred to the unit, one, as a standard, and when the signification of the term ONE is clearly apprehended, that any number, whether large or small, whether integral or fractional, may be deduced from the standard by an easy and known process.

In space, also, which is indefinite in extent, and exactly similar in all its parts, the faculties of the mind have established ideal boundaries. These boundaries give rise to the geometrical magnitudes, each of which has its own unit of measure; and by these simple contrivances, we measure space, even to the stars, as with a yardstick.

§ 161. We have, thus far, not alluded to the difficulty of *determining* the exact length of that which we regard as a standard. We are presented with a given length, and told that it is called a foot or a yard; and this being usually done at a period of life when the mind is satisfied with mere facts, we adopt the conception of a distance corresponding to a name, and then by multiplying and dividing that distance we are enabled to apprehend other distances. But this by no means answers the inquiry, What is the standard for measurement?

Under the supposition that the laws of physical nature operate uniformly, the unit of measure in England and the United States has been fixed

by ascertaining the length of a pendulum which will vibrate seconds, and to this length the Imperial yard, which we have also adopted as a standard, is referred. Hence the unit of measure is referred to a natural standard, viz. to the distance between the axis of suspension and the centre of oscillation of a pendulum which shall vibrate seconds in vacuo, in London, at the level of the sea. This distance is declared to be 39.1393 *imperial inches*; that is, 3 imperial feet and 3.1393 inches. Thus, the determination of the unit of length demands the application of the most abstruse science, combined with accurate observation and delicate experiment.

Could this distance, or unit, have been exactly ascertained before the measures of the world were fixed, and in general use, it would have afforded a standard at once certain and convenient, and all distances would then have been expressed in numbers arising from its multiplication or exact division. But as the measures of the world (and consequently their units) were fixed antecedently to the determination of this distance, it was expressed in measures already known; and hence, instead of being represented by 1, which had already been appropriated to the foot, it was expressed in terms of the foot, viz. 39.1393 inches, and this is now the standard to which all units of measure are referred.

§ 162. The French Government, in order to obtain an invariable unit, measured a degree of the arc of a meridian on the earth's surface; and from this computed the length of the meridional arc from the equator to the pole. This length they divided into ten million equal parts, and then took one of these parts for the unit of length, and called it a **METER**. The length of this meter is equal to 1 yard 3 inches and 37 hundredths of an inch, very nearly. Thus they obtained the length of the unit which is the base of the Metric System of Weights and Measures. The next step was to fix the *law* by which the other units of the system should be obtained from the base. As the scale of tens is the simplest law by which we can pass from one unit to another, that scale was adopted, and the larger units are formed by multiplying the base continually by 10, and the smaller, by dividing it continually by 10.

The Metric System of Weights and Measures, by furnishing a common language to all nations, applicable to the measure, cost, and value of every production of labor, of art, and of commerce, will, if universally adopted, go far towards bringing about entire uniformity; and will produce similar changes in the department of industry, to those which the telegraph has produced in the department of thought.

§ 163. The unit of measure, whether it be the im-

perial yard or the French meter, is not only important as affording a basis for all measurement, but is also the element from which we deduce the unit of weight. The weight of 27.7015 cubic inches of distilled water is taken as the English standard of weight, weighing exactly one pound avoirdupois, and this quantity of water is determined from the unit of length; that is, the determination of it reaches back to the length of a pendulum which will vibrate seconds in the latitude of London.

§ 164. Two geometrical figures are said to be equal, when they contain the same unit of measure an equal number of times. Two figures are said to be equal in all their parts, when they can be so applied to each other as to coincide throughout their whole extent. Hence, equal refers to measure, and equality in all their parts, to coincidence. Indeed, coincidence is the only test of perfect geometrical equality.

PROPERTIES OF FIGURES.

§ 165. A property of a figure is a mark common to all figures of the same class. For example; if the class be "Quadrilateral," there are two very obvious properties, common to all quadrilaterals, besides the one which characterizes the figure, and by which its name is defined, viz. that it has four angles,

and that it may be divided into two triangles. If the class be "Parallelogram," there are several properties common to all parallelograms, and which are subjects of proof; such as, that the opposite sides and angles are equal; the diagonals divide each other into equal parts, &c. If the class be "Triangle," there are many properties common to all triangles, besides the characteristic that each has three sides. If the class be a particular kind of triangle, such as the equilateral, isosceles, or right-angled triangle, then each class has particular properties, common to every individual of the class, but not common to the other classes.

It is important, however, to remark, that every property which belongs to "triangle," regarded as a genus, will appertain to every species or class of triangle; and universally, every property which belongs to a genus will belong to every species under it; and every property which belongs to a species will belong to every class or subspecies under it; and every property which belongs to one of a subspecies or class will be common to every individual of the class. For example: "the square on the hypotenuse of a right-angled triangle is equal to the sum of the squares described on the other two sides," is a proposition equally true of every right-angled triangle: and "every straight line perpendicular to a chord, at the middle point, will pass through the centre," is equally true of all chords and of all circles.

MARKS OF WHAT MAY BE PROVED.

§ 166. The characteristic properties of every geometrical figure (that is, those properties without which the figures could not exist), are given in the definitions. How are we to arrive at all the other properties of these figures? The propositions implied in the definitions, viz. that things corresponding to the words defined do or may exist with the properties named; and the self-evident propositions or axioms, contain the only marks of what can be proved; and by a skilful combination of these marks we are able to discover and prove all that is discovered and proved in Geometry.

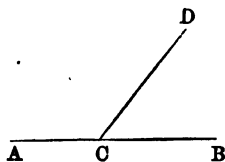
Definitions and axioms, and propositions deduced from them, are major premises in each new demonstration; and the science is made up of the processes employed for bringing unforeseen cases under these known truths; or, in syllogistic language, for proving the minors necessary to complete the syllogisms. The marks being so few, and the inductions which furnish them so obvious and familiar, there would seem to be very little difficulty in the deductive processes which follow. The connecting together of several of these marks are Deductions, or Trains of Reasoning; and hence, Geometry is a Deductive Science.

DEMONSTRATION.

§ 167. As a first example, let us take the first proposition in Legendre's Geometry:

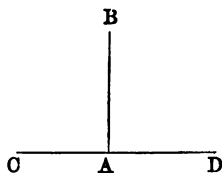
"If a straight line meet another straight line, the sum of the two adjacent angles will be equal to two right angles."

Let the straight line DC meet the straight line AB at the point C, then will the angle ACD plus the angle DCB be equal to two right angles.



To prove this proposition, we need the definition of a right angle, viz.:

When a straight line AB meets another straight line CD, so as to make the adjacent angles BAC and BAD equal to each other, each of those angles is called a RIGHT ANGLE, and the line AB is said to be PERPENDICULAR to CD.



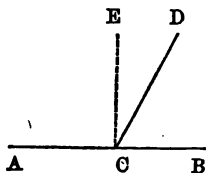
We shall also need the 1st, 2d, and 9th axioms, for inferring equality, viz.:

1. Things which are equal to the same thing are equal to each other.

2. If equals be added to equals, the sums will be equal.

9. A whole is equal to the sum of all its parts.

Now, before these formulas or tests can be applied, it is necessary to suppose a straight line CE to be drawn perpendicular to AB at the point C : then by the definition of a right angle, the angle ACE will be equal to the angle ECB .



By axiom 9th, we have,

ACD equal to ACE plus ECD : to each of these equals add DCB ; and by the 2d axiom we shall have,

ACD plus DCB equal to ACE plus ECD plus DCB ; but by axiom 9th,

ECD plus DCB equals ECB ; therefore by axiom 1st,

ACD plus DCB equals ACE plus ECB .

But by the definition of a right angle,

ACE plus ECB equals two right angles: therefore, by the 1st axiom,

ACD plus DCB equals two right angles.

It will be seen that the conclusiveness of the proof results,

1st. From the definition, that ACE and ECB are equal to each other, and each is called a right angle: consequently, their sum is equal to two right angles; and,

2d. In showing, by means of the axioms, that

ACD plus DCB equals ACE plus ECB; and then *inferring* from axiom 1st, that ACD plus DCB equals two right angles.

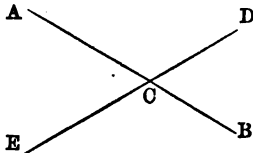
§ 168. The difficulty in the geometrical reasoning consists mainly in showing that the proposition to be proved contains the marks which prove it. To accomplish this, it is frequently necessary to draw many auxiliary lines, forming new figures and angles, which can be shown to possess marks of these marks, and which thus become connecting links between the known and the unknown truths. Indeed, most of the skill and ingenuity exhibited in the geometrical processes are employed in the use of these auxiliary means. The example above affords an illustration.

We were unable to show that the sum of the two angles possessed the mark of being equal to two right angles, until we had drawn a perpendicular, or supposed one drawn, at the point where the given lines intersect. That being done, the two right angles ACE and ECB were formed which enabled us to *compare* the sum of the angle ACD and DCB with two right angles, and thus we *proved* the proposition.

§ 169. As a second example, let us take the following proposition :

If two straight lines meet each other, the opposite or vertical angles will be equal.

Let the straight line AB meet the straight line ED at the point C: then will the angle ACD be equal to the opposite angle ECB; and the angle ACE equal to the angle DCB.



To prove this proposition, we need the last proposition, and also the 1st and 5th axioms of Legendre, viz.:

“If a straight line meet another straight line, the sum of the two adjacent angles will be equal to two right angles.”

“Things which are equal to the same thing are equal to each other.”

“If equals be taken from equals, the remainders will be equal.”

Now, since the straight line AC meets the straight line ED at the point C, we have,

ACD plus ACE equal to two right angles.

And since the straight line DC meets the straight line AB, we have,

ACD plus DCB equal to two right angles: hence, by the first axiom,

ACD plus ACE equals ACD plus DCB: taking from each the common angle ACD, we know from the fifth axiom that the remainders will be equal;

that is, the angle ACE equal to the opposite or vertical angle DCB.

§ 170. The two demonstrations given above combine all the processes of proof employed in every demonstration of the same class. When any new truth is to be proved, the known tests of truth are gradually extended to auxiliary quantities having a more intimate connection with such new truth than existed between it and the known tests, until finally, the known tests, through a series of links, become applicable to the final truth to be established: the intermediate processes, as it were, bridging over the space between the known tests and the final truth to be proved.

§ 171. There are two classes of demonstrations, quite different from each other in some respects, although the same processes of argumentation are employed in both, and although the conclusions in both are subjected to the same logical tests. They are called Direct or Positive Demonstration, and Negative Demonstration, or the *Reductio ad Absurdum*.

§ 172. The main differences in the two methods are these: The method of direct demonstration rests its arguments on known and admitted truths, and

shows by logical processes that the proposition can be brought under some previous definition, axiom, or proposition: while the negative demonstration rests its arguments on an hypothesis, combines this with *known propositions*, and deduces a conclusion by processes strictly logical. Now if the conclusion so deduced agrees with any known truth, we infer that the hypothesis (which was the only link in the chain not previously known) was true; but if the conclusion be excluded from the truths previously established; that is, if it be opposed to any one of them, then it follows that the hypothesis, being contradictory to such truth, must be false. In the negative demonstration, therefore, the *conclusion* is compared with the truths known antecedently to the proposition in question: if it agrees with any one of them, the hypothesis is correct; if it disagrees with any one of them, the hypothesis is false.

§ 173. We will give as an illustration of this method, the demonstration employed in Proposition XVII. of the First Book of Legendre:

“If two right-angled triangles have the hypotenuse and a side of the one equal to the hypotenuse and a side of the other, each to each, the triangles will be equal in all their parts.”

In the two right-angled triangles BAC and EDF

(see next figure), let the hypotenuse AC be equal to DF, the side BA to the side ED: then will the side BC be equal to EF, the angle A to the angle D, and the angle C to the angle F. To prove this proposition, we need the following, which have been before proved; viz.:

Prop. X. (of Legendre). "If two triangles have the three sides of the one equal to the three sides of the other, each to each, the triangles will be equal in all their parts."

Prop. V. "If two triangles have two sides and the included angle of the one, equal to two sides and the included angle of the other, each to each, the two triangles will be equal in all their parts."

Axiom 1. "Things which are equal to the same thing, are equal to each other."

Axiom 10 (of Legendre). "All right angles are equal."

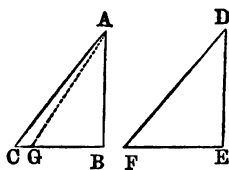
Prop. XV. "If from a point without a straight line, a perpendicular be let fall on the line, and oblique lines be drawn to different points of it,

1st. "The perpendicular will be shorter than any oblique line;

2d. "Of two oblique lines, drawn at pleasure, that which is farther from the perpendicular will be the longer."

These are the data.

Now the two sides BC and EF are either equal or unequal. If they are equal, then by Prop. X. the remaining parts of the two triangles are also equal, and the triangles themselves are equal in all their parts. If the two sides are unequal, one of them must be greater than the other: suppose BC to be the greater.



On the greater side BC take a part BG , equal to EF , and draw AG . Then, in the two triangles BAG and DEF the angle B is equal to the angle E , by axiom 10 (Legendre), both being right angles. The side AB is equal to the side DE , and by hypothesis the side BG is equal to the side EF : then it follows from Prop. V. that the side AG is equal to the side DF . But the side DF is equal to the side AC : hence, by axiom 1, the side AG is equal to AC . But the line AG cannot be equal to the line AC , having been shown to be less than it by Prop. XV.; hence, the conclusion contradicts a known truth, and is therefore false; consequently, the supposition (on which the conclusion rests), that BC and EF are unequal, is also false; therefore, as they cannot be unequal, they are equal.

§ 174. It is often supposed, though erroneously, that the Negative Demonstration, or the demonstra-

tion

tion involving the "reductio ad absurdum," is less conclusive and satisfactory than direct or positive demonstration. This impression is simply the result of a want of proper analysis.

For example: in the demonstration just given, it was proved that the two sides BC and EF cannot be *unequal*; for, such a supposition, in a logical argumentation, resulted in a conclusion directly opposed to a known truth; and as equality and inequality are the only general conditions of relation between two quantities, it follows that if they do not fulfil the one, they *must* the other. In both kinds of demonstration, the premises and conclusion agree; that is, they are both true, or both false; and the reasoning or argument, in both, is supposed to be strictly logical.

In the direct demonstration, the premises are known, being antecedent truths; and hence, the conclusion is true. In the negative demonstration, *one element* is assumed, and the conclusion is then compared with truths previously established. If the conclusion is found to agree with any one of these, we infer that the hypothesis or assumed element is true; if it contradicts any one of these truths, we infer that the assumed element is false, and hence that its opposite is true.

§ 175. Having explained the meaning of the term

Measured, as applied to a geometrical magnitude, viz. that it implies the comparison of a magnitude with its unit of measure; and having also explained the signification of the word Property, and the processes of reasoning by which, in all figures, properties not before noticed are inferred from those that are known; we shall now add a few remarks on the relations of the geometrical figures, and the methods of comparing them with each other.

PROPORTION OF FIGURES.

§ 176. Proportion is the relation which one geometrical magnitude bears to another of the same kind, with respect to its being greater or less. The two magnitudes so compared are called terms, and the measure of the proportion is the quotient which arises from dividing the second term by the first, and is called their Ratio. Only quantities of the same kind can be compared together, and it follows from the nature of the relation that the quotient, or ratio, of any two terms will be an abstract number, whether the terms themselves be abstract or concrete.

§ 177. The term Proportion is defined by most authors, "An equality of ratios between four numbers or quantities, compared together two and two."

A proportion certainly arises from such a comparison: thus, if

$$\frac{B}{A} = \frac{D}{C}; \text{ then,}$$

$$A : B :: C : D$$

is a proportion.

But if we have two quantities A and B, which may change their values, and are, at the same time, so connected together that one of them shall increase or decrease just as many times as the other, their ratio will not be altered by such changes; and the two quantities are then said to be proportional.

Thus, if A be increased three times and B three times, then,

$$\frac{3B}{3A} = \frac{A}{B};$$

that is, 3 A and 3 B bear to each other the same proportion, as A and B. Science needs a general term to express this relation between *two* quantities which change their values, without altering their quotient, and the term "proportional," or "in proportion," is employed for that purpose.

There is a symbol or character to express the relation between two quantities, when they undergo changes of value, without altering their ratio. That

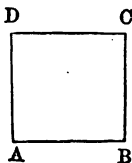
character is $\propto$, and is read "proportional to." Thus, if we have two quantities denoted by A and B, written,

$$A \propto B,$$

the expression is read, "A proportional to B."

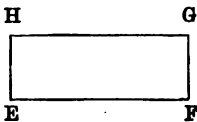
§ 178. There is yet another kind of relation which may exist between two quantities A and B, which it is very important to consider and understand. Suppose the quantities to be so connected with each other, that when the first is increased according to any law of change, the second shall decrease according to the same law; and the reverse.

For example: the area of a rectangle is equal to the product of its base and altitude. Then, in the rectangle ABCD, we have



$$\text{Area} = AB \times BC.$$

Take a second rectangle EFGH, having a longer base EF, and a less altitude FG, but such that it shall have an equal area with the first: then we shall have



$$\text{Area} = EF \times FG.$$

Now since the areas are equal, we shall have

$$AB \times BC = EF \times FG;$$

and by resolving the terms of this equation into a proportion, we shall have

$$AB : EF :: FG : BC.$$

It is plain that the sides of the rectangle ABCD may be so changed in value as to become the sides of the rectangle EFGH, and that while they are undergoing this change, AB will increase and BC diminish. The change in the values of these quantities will therefore take place according to a fixed law: that is, one will be diminished as many times as the other is increased, since their product is constantly equal to the area of the rectangle EFGH.

Denote the side AB by x and BC by y , and the area of the rectangle EFGH, which is known, by a ; then

$$xy = a;$$

and when the product of two varying quantities is constantly equal to a known quantity, the two quantities are said to be *Reciprocally* or *Inversely* proportional; thus x and y are said to be inversely proportional to each other. If we divide 1 by each member of the above equation, we shall have

$$\frac{1}{xy} = \frac{1}{a};$$

and by multiplying both members by x , we shall have

$$\frac{1}{y} = \frac{x}{a};$$

and then by dividing both numbers by x , we have

$$\frac{\frac{1}{y}}{x} = \frac{1}{a};$$

that is, the ratio of x to $\frac{1}{y}$ is constantly equal to $\frac{1}{a}$; that is, equal to the same quantity, however x or y may vary; for, a and consequently $\frac{1}{a}$ does not change. Hence,

Two quantities, which may change their values, are reciprocally or inversely proportional, when one is proportional to unity divided by the other, and then their product remains constant.

We express this reciprocal or inverse relation thus :

$$A \propto \frac{1}{B}.$$

A is said to be inversely proportional to B: the symbols also express that A is directly proportional to $\frac{1}{B}$. If we have

$$A \propto \frac{B}{C},$$

we say, that A is directly proportional to B, and inversely proportional to C.

The terms Direct, Inverse or Reciprocal, apply to

the *nature* of the *proportion*, and not to the Ratio, which is always a mere quotient and the measure of proportion. The term Direct applies to all proportions in which the terms increase or decrease together; and the term Inverse or Reciprocal to those in which one term increases as the other decreases. They cannot, therefore, properly be applied to ratio without changing entirely its signification and definition.

COMPARISON OF FIGURES.

§ 179. In comparing geometrical magnitudes, by means of their quotient, it is not the quotient alone which we consider. The comparison implies a general relation of the magnitudes, which is measured by the Ratio. For example: we say that "Similar triangles are to each other as the squares of their homologous sides." What do we mean by that? Just this:

That the area of a triangle
Is to the area of a similar triangle,
As the area of a square described on a side of the
first
To the area of a square described on a homologous side of the second.

Thus, we see that every term of such a proportion is in fact a surface, and that the area of a triangle increases or decreases much faster than its sides; that is,

if we double each side of a triangle, the area will be four times as great : if we multiply each side by three, the area will be nine times as great ; or if we divide each side by two, we diminish the area four times, and so on. Again :

The area of one circle

Is to the area of another circle,

As a square described on the diameter of the first

To a square described on the diameter of the second.

Hence, if we double the diameter of a circle, the area of the circle whose diameter is so increased will be increased four times ; if we multiply the diameter by three, the area will be increased nine times ; and similarly, if we divide the diameter.

§ 180. In comparing volumes together, the same general principles obtain. Similar volumes are to each other as the cubes described on their homologous or corresponding sides. That is,

A prism

Is to a similar prism,

As a cube described on a side of the first

Is to a cube described on a homologous side of the second.

Hence, if the sides of a prism be doubled, the volumes, or contents, will be increased eight-fold. Again :

A sphere

Is to a sphere,

As a cube described on the diameter of the first

Is to a cube described on a diameter of the second.

Hence, if the diameter of a sphere be doubled, its volume, or contents, will be increased eight-fold ; if the diameter be multiplied by three, the volume, or contents, will be increased twenty-seven fold ; if the diameter be multiplied by four, the volume, or contents, will be increased sixty-four fold ; the volumes increasing as the cubes of the numbers 1, 2, 3, 4, &c.

§ 181. The relation or ratio of two magnitudes to each other, may be, and indeed is, expressed by an abstract number. This number has a fixed value so long as we do not introduce a change in the volumes ; but, if we wish to express their ratio under the supposition that their volumes may change according to fixed laws (that is, so that the figures shall continue similar), we then compare them with similar figures described on their homologous or corresponding sides ; or, what is the same thing, take into account the corresponding changes which take place in the abstract numbers that express their volumes.

RECAPITULATION.

§ 182. We have now completed a general outline of the science of Geometry, and what has been said may be recapitulated under the following heads. It has been shown,

1st. That Geometry is conversant about space, or those limited portions of space which are called, Geometrical Magnitudes.

2d. That the geometrical magnitudes embrace four species of figures :

1st. Lines—straight and curved ;

2d. Surfaces—plane and curved ;

3d. Volumes—bounded either by plane surfaces or curved, or both ; and,

4th. Angles, arising from the positions of lines and planes, and by which they are bounded.

3d. That the science of Geometry is made up of those processes by means of which all the properties of these magnitudes are examined and developed, and that the results arrived at constitute the truths of Geometry.

4th. That the truths of Geometry may be divided into three classes ;

1st. Those implied in the definitions, viz., that things exist corresponding to certain words defined ;

2d. Intuitive or self-evident truths embodied in the axioms ;

3d. Truths deduced (that is, inferred) from the definitions and axioms, called Demonstrative Truths.

5th. That the examination of the properties of the geometrical magnitudes has reference,

1st. To their comparison with a standard or unit of measure ;

2d. To the discovery of properties belonging to an individual figure, and yet common to the entire class to which such figure belongs ;

3d. To the comparison, with respect to magnitude, of figures of the same species with each other ; viz. lines with lines, surfaces with surfaces, volumes with volumes, and angles with angles.

SUGGESTIONS FOR THOSE WHO TEACH GEOMETRY.

1. Be sure that your pupils have a clear apprehension of space, and of the notion that Geometry is conversant about space only.

2. Be sure that they understand the signification of the terms, lines, surfaces, volumes, and angles, and that these names indicate certain portions of space corresponding to them.

3. See that they understand the distinction between a straight line and a curve ; between a plane surface and a curved surface ; between a volume bounded by

planes and a volume bounded by curved surfaces ; and also, between the different kinds of angles.

4. Be careful to have them note the characteristics of the different species of plane figures, such as triangles, quadrilaterals, pentagons, hexagons, &c. ; and then the characteristic of each class or subspecies, so that the name shall recall, at once, the characteristic properties of each figure.

5. Be careful, also, to have them note the characteristic differences of the volumes. Let them often name and distinguish those which are bounded by planes, those bounded by plane and curved surfaces, and those bounded by curved surfaces only. Regarding Volumes as a genus, let them give the species and subspecies into which the volumes may be divided.

6. Having thus made them familiar with the things which are the subjects of the reasoning, explain carefully the nature of the definitions : then of the axioms, the grounds of our belief in them, and the sources from which those self-evident truths are inferred.

7. Then explain to them, that the definitions and axioms are the basis of all geometrical reasoning : that every proposition must be deduced from them, and that they afford the tests of all the truths which the reasonings establish.

8. Let every figure, used in a demonstration, be accurately drawn, by the pupil himself, on a black-board. This will establish a connection between the

eye and the hand, and give, at the same time, a clear perception of the figure and a distinct apprehension of the relation of its parts.

9. Let the pupil, in every demonstration, first enunciate, in general terms, that is, without the aid of a diagram, or any reference to one, the proposition to be proved; and then state the principles previously established, which are to be employed in making out the proof.

10. When, in the course of a demonstration, any truth is inferred from its connection with one before known, let the truth so referred to be fully and accurately stated, even though the number of the proposition in which it is proved be also required. This is deemed important.

11. Let the pupil be made to understand that a demonstration is but a series of logical arguments arising from comparison, and that the result of every comparison, in respect to quantity, contains the mark either of equality or inequality.

12. Let the distinction between a positive and negative demonstration be fully explained and clearly apprehended.

13. In the comparison of quantities with each other, great care should be taken to impress the fact that proportion exists only between quantities of the same kind, and that ratio is the measure of proportion.

14. Do not fail to give much importance to the

kind of quantity under consideration. Let the question be often put, What kind of quantity are you considering? Is it a line, a surface, a volume, or an angle? And what kind of a line, surface, volume or angle?

15. In all cases of measurement, the unit of measure should receive special attention. If lines are measured, or compared by means of a common unit, see that the pupil perceives that unit clearly, and apprehends distinctly its relations to the lines which it measures. In surfaces, take much pains to mark out on the black-board the particular square which forms the unit of measure, and write unit, or unit of measure, over it. So in the measurement of volumes, let the unit or measuring cube be exhibited, and the conception of its size clearly formed in the mind; and then impress the important fact, that, *all measurement consists in merely comparing a unit of measure with the quantity measured; and that the number which expresses the ratio is the numerical expression for that measure.*

16. Be careful to explain the difference of the terms *Equal* and *Equal in all the parts*, and never permit the pupil to use them as synonymous. An accurate use of words leads to nice discriminations of thought.

SECTION V.

ANALYSIS.

§ 183. ANALYSIS is a general term, embracing that entire portion of mathematical science in which the quantities considered are represented by letters of the alphabet, and the operations to be performed on them are indicated by signs.

§ 184. We have seen that all numbers must be numbers of something; for, there is no such thing as a number without a base: that is, every number must be based on the abstract unit one, or on some unit denominated. But although numbers must be numbers of *something*, yet they may be numbers of *any thing*, for the unit may be whatever we choose to make it.

§ 185. All quantity consists of parts, which can be numbered exactly or approximatively, and, in this respect, possesses all the properties of number. Propositions, therefore, concerning numbers, have the re-

markable peculiarity, that they are propositions concerning all quantities whatever. That half of six is three, is equally true, whatever the word six may represent, whether six abstract units, six men, or six triangles. Analysis extends the generalization still further. A number represents, or stands for, that particular number of things of the same kind, without reference to the *nature* of the thing; but an analytical symbol does more, for it may stand for *all numbers*, or for all quantities which numbers represent, or even for quantities which cannot be exactly expressed numerically.

As soon as we conceive of a thing we may conceive it divided into equal parts, and may represent either or all of those parts by a or x , or may, if we please, denote the thing itself by a or x , without any reference to its being divided into parts.

§ 186. In Geometry, each geometrical figure stands for a class; and when we have demonstrated a property of a figure, that property is considered as proved for every figure of the class. For example: when we prove that the square described on the hypotenuse of a right-angled triangle is equal to the sum of the squares described on the other two sides, we demonstrate the fact for *all right-angled triangles*. But in analysis, all numbers, all lines, all surfaces, all volumes, and all angles, may be denoted by a single

symbol, a or x . Hence, all truths inferred by means of these symbols are true of all things whatever, and not, like those of number and geometry, true only of particular classes of things. It is, therefore, not surprising, that the symbols of analysis do not excite in our minds the ideas of particular things. The mere written characters, a , b , c , d , x , y , z , serve as the representatives of things in general, whether abstract or concrete, whether known or unknown, whether finite or infinite.

§ 187. In the uses which we make of these symbols, and the processes of reasoning carried on by means of them, the mind insensibly comes to regard them as *things*, and not as mere signs; and we constantly predicate of them the properties of things in general, without pausing to inquire what kind of thing is implied. Thus, we define an equation to be a proposition in which equality is predicated of one thing as compared with another. For example:

$$a + c = x,$$

is an equation, because x is declared to be equal to the sum of a and c . In the solution of equations, we employ the axioms, "If equals be added to equals, the sums will be equal;" and, "If equals be taken from equals, the remainders will be equal." Now,

these axioms do not express qualities of language, but properties of quantity. Hence, all inferences in mathematical science, deduced through the instrumentality of symbols, whether Arithmetical, Geometrical, or Analytical, must be regarded as concerning quantity, and not symbols.

Since analytical symbols are the representatives of quantity in general, there is no necessity of keeping the idea of *quantity* continually alive in the mind; and the processes of thought may, without danger, be allowed to rest on the symbols themselves, and therefore become, to that extent, merely mechanical. But, when we look back and see on what the reasoning is based, and how the processes have been conducted, we shall find that every step was taken on the supposition that we were actually dealing with things, and not symbols; and that, without this understanding of the language, the whole system is without signification, and fails.

§ 188. There are four principal branches of Analysis:

1st. Algebra.

2d. Analytical Geometry.

3d. Analytical Trigonometry.

4th. Differential and Integral Calculus.

ALGEBRA.

§ 189. Algebra is, in fact, a species of universal Arithmetic, in which letters and signs are employed to abridge and generalize all processes involving numbers. It is divided into two parts, corresponding to the science and art of Arithmetic:

1st. That which has for its object the investigation of the properties of numbers, embracing all the processes of reasoning by which new properties are inferred from known ones; and,

2d. The solution of all problems or questions involving the determination of certain numbers which are unknown, from their connection with certain others which are known or given.

ANALYTICAL GEOMETRY.

§ 190. Analytical Geometry examines the properties, measures, and relations of the geometrical magnitudes by means of the analytical symbols. This branch of mathematical science originated with the illustrious Descartes, a celebrated French mathematician of the seventeenth century. He observed that the positions of points, the direction of lines, and the forms of surfaces, could be expressed by means of the algebraic symbols; and consequently,

that every change, either in the position or extent of a geometrical magnitude, produced a corresponding change in certain symbols, by which such magnitude could be represented. As soon as it was found that, to every variety of position in points, direction in lines, or form of curves or surfaces, there corresponded certain analytical expressions (called their equations), it followed, that if the processes were known by which these equations could be examined, the relation of their parts determined, and the laws according to which those parts vary, relative to one another, ascertained, then the corresponding changes in the geometrical magnitudes, thus represented, could be immediately inferred.

Hence, it follows that every geometrical question can be solved, if we can resolve the corresponding algebraic equation; and the power over the geometrical magnitudes was extended just in proportion as the properties of quantity were brought to light by means of the Calculus. The applications of this Calculus were soon extended to the subjects of mechanics, astronomy, and indeed, in a greater or less degree, to all branches of natural philosophy; so that, at the present time, all the varieties of physical phenomena, with which the higher branches of the science are conversant, are found to answer to varieties determinable by the algebraic analysis.

Two classes of quantities, and consequently two sets of symbols, quite distinct from each other, enter into this Calculus; the one called *Constants*, which preserve a fixed or given value throughout the same discussion or investigation; and the other called *Variables*, which undergo certain changes of value, the laws of which are indicated by the algebraic expressions or equations, into which they enter. Hence,

Analytical Geometry may be defined as that branch of mathematical science which examines, discusses, and develops the properties of the geometrical magnitudes by noting the changes that take place in the algebraic symbols which represent them, the laws of change being determined by an algebraic equation or formula.

ANALYTICAL TRIGONOMETRY.

§ 191. Analytical Trigonometry is that branch of Mathematics which treats of the general properties and relations of the sides and angles of Triangles, by means of Analysis.

DIFFERENTIAL AND INTEGRAL CALCULUS.

§ 192. In this branch of mathematical science, as in Analytical Geometry, two kinds of quantity are considered, viz. Variables and Constants; and, con-

sequently, two distinct sets of symbols are employed. The science consists of a series of processes which note the changes that take place in the values of the Variables. Those changes of value take place according to fixed laws established by algebraic formulas, and are indicated by certain *marks* drawn from the variable symbols, called *Differential Coefficients*. By these marks we are enabled to trace out, with the accuracy of exact science, the most hidden properties of quantity, as well as the most general and minute laws which regulate its changes of value.

§ 193. It will be observed, that Analytical Geometry and the Differential and Integral Calculus treat of quantity regarded under the same general aspect, viz. as subject to changes or variations in magnitude, according to laws indicated by algebraical formulas; and the quantities, whether variable or constant, are, in both cases, represented by the same algebraic symbols, viz. the constants by the first, and the variables by the final letters of the alphabet. There is, however, this important difference: in Analytical Geometry all the results are inferred from the relations which exist between the quantities themselves, while in the Differential and Integral Calculus they are deduced by considering what may be indicated by *marks* drawn from variable quantities, under certain suppositions, and by *marks of such marks*.

§ 194. Algebra, Analytical Geometry, the Differential and Integral Calculus, extended into the Theory of Variations, make up the subject of analytical science, of which Algebra is the elementary branch. As the limits of this work do not permit us to discuss the subject in full, we shall confine ourselves to Algebra, pointing out, occasionally, a few of the more obvious connections between it and the two other branches.

SECTION VI.

ALGEBRA.

§ 195. On an analysis of the subject of Algebra, we think it will appear that the subject itself presents no serious difficulties, and that most of the embarrassment which is experienced by the pupil in gaining a knowledge of its principles, as well as in their applications, arises from not attending sufficiently to the *language* or *signs* of the thoughts which are combined in the reasonings. At the hazard, therefore, of being a little diffuse, I shall begin with the very elements of the algebraic language, and explain, with much minuteness, the exact signification of the characters that stand for the quantities which are the subjects of the analysis; and also, of those signs which indicate the several operations to be performed on the quantities.

§ 196. The Quantities which are the subjects of the algebraic analysis may be divided into two classes: those which are known or given, and those which are

unknown or sought. The known are uniformly represented by the first letters of the alphabet, *a, b, c, d*, &c.; and the unknown, by the final letters, *x, y, z, v, w*, &c.

§ 197. Quantity is susceptible of being increased or diminished and measured; and there are six operations which can be performed upon a quantity that will give results differing from the quantity itself, viz.:

- 1st. To add it to itself or to some other quantity;
- 2d. To subtract some other quantity from it;
- 3d. To multiply it by a number;
- 4th. To divide it;
- 5th. To raise it to any power;
- 6th. To extract a root of it.

The cases in which the multiplier or divisor is 1, are of course excepted; as also the cases where a root is to be extracted of 1, or 1 raised to any power.

§ 198. The six signs which denote these operations are too well known to be repeated here. These, with the signs of equality and inequality, the letters of the alphabet and the figures which are employed, make up the elements of the algebraic language. The words and phrases of the algebraic, like those of every other language, are taken in connection with each other, and are not to be interpreted as separate and isolated symbols.

§ 199. The symbols of quantity are designed to represent quantity in general, whether abstract or concrete, whether known or unknown; and the signs which indicate the operations to be performed on the quantities are to be interpreted in a sense equally general. When the sign plus is written, it indicates that the quantity before which it is placed is to be added to some other quantity; and the sign minus implies the existence of a minuend, from which the subtrahend is to be taken. One thing should be observed in regard to the signs which indicate the operations that are to be performed on quantities, viz. *they do not at all affect or change the nature of the quantity before or after which they are written, but merely indicate what is to be done with the quantity.* In Algebra, for example, the minus sign merely indicates that the quantity before which it is written is to be subtracted from some other quantity; and in Analytical Geometry, that the line before which it falls is estimated in a contrary direction from that in which it would have been reckoned, had it had the sign plus; but in neither case is the *nature* of the quantity itself different from what it would have been, had it had the sign plus.

The interpretation of the language of Algebra is the first thing to which the attention of a pupil should be directed; and he should be drilled on the meaning and import of the symbols, until their sig-

nifications and uses are as familiar as the sounds and combinations of the letters of the alphabet.

§ 200. Beginning with the elements of the language, let any number or quantity be designated by the letter a , and let it be required to add this letter to itself, and find the result, or sum. The addition will be expressed by

$$a + a = \text{the sum.}$$

But how is the sum to be expressed? By simply regarding a as *one* a , or $1a$, and then observing that one a and *one* a , make *two* a 's or $2a$: hence,

$$a + a = 2a;$$

and thus we place a figure before a letter to indicate how many times it is taken. Such figure is called a *Coefficient*.

§ 201. The product of several numbers is indicated by the sign of multiplication, or by simply writing the letters which represent the numbers by the side of each other. Thus,

$$a \times b \times c \times d \times f, \text{ or } abcd f,$$

indicates the continued product of a , b , c , d , and f ; and each letter is called a factor of the product:

hence, a *factor* of a product is one of the multipliers which produce it. Any figure, as 5, written before a product, as

$$5\ a\ b\ c\ d\ f,$$

is the coefficient of the product, and shows that the product is taken 5 times.

§ 202. If the numbers represented by a , b , c , d , and f were equal to each other, they would each be represented by a single letter a , and the product would then become

$$a \times a \times a \times a \times a = a^5;$$

that is, we indicate the product of several equal factors by simply writing the letter once and placing a figure above and a little at the right of it, to indicate how many times it is taken as a factor. The figure so written is called an *exponent*. Hence, an exponent is a simple form of expression, to point out how many equal factors are employed.

§ 203. To denote that any quantity denoted by a , is to be raised to any power, as the fifth, for example, we merely write it with an exponent 5; thus,

a^5 ; which is read, a to the fifth power.

§ 204. The division of one quantity by another is indicated by simply writing the divisor below the dividend, after the manner of a fraction ; by placing it on the right of the dividend with a horizontal line and two dots between them ; or by placing it on the right with a vertical line between them : thus either form of expression,

$$\frac{b}{a}, b \div a, \text{ or, } b | a,$$

indicates the division of b by a .

§ 205. The extraction of a root is indicated by the sign $\sqrt{}$. This sign, when used by itself, indicates the lowest root, viz. the square root. If any other root is to be extracted, as the third, fourth, fifth, &c., the figure marking the degree of the root is written above and at the left of the sign ; as,

$\sqrt[3]{}$ denotes the cube root ; $\sqrt[4]{}$, fourth root, &c.

The figure so written, is called the *Index* of the root.

We have thus given the very simple and general language by which we indicate each of the six operations that may be performed on an algebraic quantity, and *every process in Algebra* involves one or other of these operations.

MINUS SIGN.

§ 206. The algebraic symbols are divided into two classes entirely distinct from each other, viz. the symbols which represent quantities, and the signs which denote operations. We have seen that all the algebraic processes are comprised under addition, subtraction, multiplication, division, raising of powers, and the extraction of roots; and it is plain, that the *nature* of a quantity is not at all changed by prefixing to it the sign which indicates either of these operations. The quantity denoted by the letter a , for example, is the same, in *every respect*, whatever sign may be prefixed to it; that is, whether it be added to another quantity, subtracted from it, whether multiplied or divided by any number, or whether we extract the square or cube or any other root of it. The algebraic signs, therefore, must be regarded merely as indicating *operations* to be performed on quantity, and not as affecting the *nature* of the quantities to which they may be prefixed. We say, indeed, that quantities are plus and minus, but this is an abbreviated language to express that they are to be added or subtracted.

§ 207. In Algebra, as in Arithmetic and Geometry, all the principles of the science are deduced from the definitions and axioms; and the rules for performing

the operations are but directions framed in conformity to such principles. Having, for example, fixed by definition the power of the minus sign, viz. that any quantity before which it is written, shall be regarded as to be subtracted from another quantity, we wish to discover the process of performing that subtraction, so as to deduce therefrom a general *principle*, from which we can frame a rule applicable to all similar cases.

SUBTRACTION.

§ 208. Let it be required, for example, to subtract from b the difference between a and c . Now, having written the letters, with their proper signs, the language of Algebra expresses that it is the *difference* only between a and c , which is to be taken from b ; and if this difference were known, we could make the subtraction at once. But the nature and generality of the algebraic symbols, enable us to *indicate operations*, merely, and we cannot in general make reductions until we come to the final result. In what general way, therefore, can we indicate the true difference?

If we indicate the subtraction of a from b , we have $b - a$; but then, we have taken away too much from b by the number of units in c , for it was not a , but the *difference* between a and c that was to be subtracted from b . Having

$$\left| \begin{array}{r} b \\ a - c \end{array} \right|$$

$$\left| \begin{array}{r} b - a \\ b - a + c \end{array} \right|$$

taken away *too much*, the remainder is *too small* by c : hence, if c be added, the true remainder will be expressed by $b - a + c$.

Now, by analyzing this result, we see that the sign of every term of the subtrahend has been changed; and what has been shown with respect to these quantities is equally true of all others standing in the same relation: hence, we have the following general rule for the subtraction of algebraic quantities:

Change the sign of every term of the subtrahend, or conceive it to be changed, and then unite the quantities as in addition.

MULTIPLICATION.

§ 209. Let us now consider the case of multiplication, and let it be required to multiply $a - b$ by c . The algebraic language expresses that the difference between a and b is to be taken as many times as there are units in c . If we knew this difference, we could at once perform the multiplication. But by what general process is it to be performed without finding that difference? If we take a , c times, the product will be ac : but as it was only the *difference* between a and b , that was to be multiplied by c ; therefore, this product ac will be too great by

$$\begin{array}{r|l} a-b & \\ c & \\ \hline ac-bc & \end{array}$$

b taken c times; that is, the true product will be expressed by $ac-bc$. Hence, we see, that,

If a quantity having a plus sign be multiplied by another quantity having also a plus sign, the sign of the product will be plus; and if a quantity having a minus sign be multiplied by a quantity having a plus sign, the sign of the product will be minus.

§ 210. Let us now take the most general case, viz. that in which it is required to multiply $a-b$ by $c-d$.

Let us again observe that the algebraic language denotes that $a-b$ is to be taken as many times as there are units in $c-d$; and if these two differences were known, their product would at once form the product required.

$a-b$	$a-b$
$c-d$	$c-d$
	$ac-bc$
	$-ad+bd$
	$ac-bc-ad+bd$

First: let us take $a-b$ as many times as there are units in c ; this product, from what has already been shown, is equal to $ac-bc$. But since the multiplier is not c , but $c-d$, it follows that this product is too large by $a-b$ taken d times; that is, by $ad-bd$: hence, the first product diminished by this last, will give the true product. But, by the rule for subtraction, this difference is found by changing the signs of the subtrahend, and then uniting all the terms as

in addition : hence, the true product is expressed by $ac - bc - ad + bd$.

By analyzing this result, and employing an abbreviated language, we have the following general principle to which the signs conform in multiplication, viz. :

Plus multiplied by plus gives plus in the product ; plus multiplied by minus gives minus ; minus multiplied by plus gives minus ; and minus multiplied by minus gives plus in the product.

§ 211. The remark is often made by pupils that the above reasoning appears very satisfactory so long as the quantities are presented under the above form ; but why will $-b$ multiplied by $-d$ give plus bd ? How can the product of two negative quantities *standing alone* be plus ?

In the first place, the minus sign being prefixed to b and d , shows that in an *algebraic sense* they do not stand by themselves, but are connected with other quantities : and if they are not so connected, the minus sign makes no difference ; for, it in no case affects the quantity, but merely points out a connection with other quantities. Besides, the product determined above, being independent of any particular value attributed to the letters a , b , c , and d , must be of such a form as to be true for all values ; and hence for the case in which a and c are both

equal to zero. Making this supposition, the product reduces to the form of $+bd$.

The rules for the signs in division are readily deduced from the definition of division, and the principles already laid down.

ZERO AND INFINITY.

§ 212. The terms zero and infinity have given rise to much discussion, and been regarded as presenting difficulties not easily removed. It may not be easy to frame a form of language that shall convey to a mind, but little versed in mathematical science, the precise ideas which these terms are designed to express; but we are unwilling to suppose that the ideas themselves are beyond the grasp of an ordinary intellect. The terms are used to designate the *two limits* of each of the quantities, *Space*, *Number*, and *TIME*.

§ 213. Assuming any two points in space, and joining them by a straight line, the distance between the points will be truly indicated by the length of this line, and this length may be expressed numerically by the number of times which the line contains a known unit.

If, now, the points are made to approach each other, the length of the line will diminish as the

points come nearer and nearer together, until at length, when the two points become one, the length of the line will disappear, having attained its *limit*, which is called *zero*.

If, on the contrary, the points recede from each other, the length of the line joining them will continually increase; but so long as the length of the line can be expressed in terms of a known unit of measure, it is not infinite. But, if we suppose the points removed, so that any known unit of measure would occupy no *appreciable portion* of the line, then the length of the line is said to be *Infinite*.

§ 214. Assuming one as the unit of number, and admitting the self-evident truth that it may be increased or diminished, we shall have no difficulty in understanding the import of the terms zero and infinity, as applied to number. For, if we suppose the unit one to be continually diminished, by division or otherwise, the fractional units thus arising will be less and less; and in proportion as we continue the divisions, they will continue to diminish. Now, the limit or boundary to which these very small fractions approach, is called *Zero*, or nothing. So long as the fractional number forms an appreciable part of one, it is not zero, but a finite fraction; and the term zero is only applicable to that which forms no appreciable part of the standard.

If, on the other hand, we suppose a number to be continually increased, the relation of this number to the unit will be constantly changing. So long as the number can be expressed in terms of the unit one, it is finite, and not infinite; but when the unit one forms no *appreciable* part of the number, the term *infinite* is used to express that state of value, or, rather, that limit of value.

§ 215. The terms zero and infinity are therefore employed to designate the limits to which decreasing and increasing quantities may be made to approach nearer than any assignable quantity; but these limits cannot be compared, in respect to magnitude, with any known standard, so as to give a finite ratio.

§ 216. It may, perhaps, appear somewhat paradoxical, that zero and infinity should be defined as "the limits of number and space" when they are in themselves not measurable. But a limit is that "which sets bounds to, or circumscribes;" and as all finite space and finite number (and such only are implied by the terms Space and Number), are contained between zero and infinity, we employ these terms to designate the limits of Number and Space.

OF THE EQUATION.

§ 217. We have seen that all deductive reasoning involves certain processes of comparison, and that the syllogism is the formula to which those processes may be reduced. It has also been stated that if two quantities be compared together, there will necessarily result the condition of equality or inequality. The equation is an analytical formula for expressing equality.

§ 218. The subject of equations is divided into two parts. The first, consists in finding the equation; that is, in the process of expressing the relations existing between the quantities considered, by means of the algebraic symbols and formula. This is called the Statement of the proposition. The second is purely deductive, and consists, in Algebra, in what is called the solution of the equation, or finding the value of the unknown quantity: and in the other branches of analysis, it consists in the discussion of the equation; that is, in the drawing out from the equation every thing which it is capable of expressing.

§ 219. Making the statement, or finding the equation, is merely analyzing the problem, and expressing its elements and their relations in the language of analysis. It is, in truth, collating the facts, noting their

bearing and connection, and inferring some general law or principle which leads to the formation of an equation.

The condition of equality between two quantities is expressed by the sign of equality, which is placed between them. The quantity on the left of the sign of equality is called the first member, and that on the right, the second member of the equation. The first member corresponds to the subject of a proposition; the sign of equality is copula and part of the predicate, signifying, IS EQUAL TO. Hence, an equation is merely a proposition expressed algebraically, in which equality is predicated of one quantity as compared with another. It is the great formula of analysis.

§ 220. We have seen that every quantity is either abstract or denominate; hence, an equation, which is a general formula for expressing equality, must be either abstract or denominate.

An abstract equation expresses merely the relation of equality between two abstract quantities: thus,

$$a + b = x,$$

is an abstract equation, if no unit of value be assigned to either member; for, until that be done the abstract unit one is understood, and the formula merely ex-

presses that the sum of a and b is equal to x , and is true, equally, of all quantities.

But if we assign a denominate unit of value, that is, say that a and b shall each denote so many pounds weight, or so many feet or yards of length, x will be of the same denomination, and the equation will become denominate.

§ 221. We have seen that there are six operations which may be performed on an algebraic quantity. We assume, as an axiom, that if the same operation, under either of these processes, be performed on both members of an equation, the equality of the members will not be changed. Hence, we have the six following

AXIOMS.

1. If equal quantities be added to both members of an equation, the equality of the members will not be destroyed.

2. If equal quantities be subtracted from both members of an equation, the equality will not be destroyed.

3. If both members of an equation be multiplied by the same number, the equality will not be destroyed.

4. If both members of an equation be divided by

the same number, the equality will not be destroyed.

5. If both members of an equation be raised to the same power, the equality of the members will not be destroyed.

6. If the same root of both members of an equation be extracted, the equality of the members will not be destroyed.

Every operation performed on an equation will fall under one or other of these axioms, and they afford the means of solving all equations which admit of solution.

§ 222. The term Equality, in Geometry, expresses that two magnitudes have equal measures; that is, that they contain the same unit an equal number of times. It has the same signification in Algebra.

GENERAL REMARKS.

§ 223. We have thus pointed out some of the marked characteristics of analysis. In Algebra, the elementary branch, the quantities, about which the science is conversant, are divided, as has been already remarked, into known and unknown, and the connections between them, expressed by the equation, afford the means of tracing out further

relations, and of finding the values of the unknown quantities, in terms of the known.

In the other branches of analysis, the quantities considered are divided into two general classes, Constant and Variable; the former preserving fixed values throughout the same process of investigation, while the latter undergo changes of value according to fixed laws; and from such changes we deduce, by means of the equation, common principles, and general properties applicable to all quantities.

§ 224. The correspondence between the processes of reasoning, as exhibited in the subject of general logic, and those which are employed in mathematical science, is readily accounted for, when we reflect, that the reasoning process is essentially the same in all cases; and that any change in the language employed, or in the subject to which the reasoning is applied, does not at all change the nature of the process, or materially vary its form.

§ 225. We shall not pursue the subject of analysis any further; for, it would be foreign to the purposes of the present work to attempt more than to point out the general features and characteristics of the different branches of mathematical science. We have aimed only to present the subjects about which

the science is conversant, to explain the peculiarities of the language, the nature of the reasoning processes employed, and of the connecting links of that golden chain which binds together all the parts, forming an harmonious whole.

SUGGESTIONS FOR THOSE WHO TEACH ALGEBRA.

1. Be careful to explain that the letters employed, are the mere symbols of quantity. That of, and in themselves, they have no meaning or signification whatever, but are used merely as the signs or representatives of such quantities as they may be employed to denote.

2. Be careful to explain that the signs which are used are employed merely for the purpose of indicating the six operations which may be performed on quantity; and that they indicate operations merely, without at all affecting the nature of the quantities before which they are placed.

3. Explain that the letters and signs are the elements of the algebraic language, and that the symbolical language itself arises from the combination of these elements.

4. Explain that the finding of an algebraic formula is but the translation of certain ideas, first expressed in our common language, into the language

of Algebra; and that the interpretation of an algebraic formula is merely translating its various significations into common language.

5. Let the language of Algebra be carefully studied, so that its construction and significations may be clearly apprehended.

6. Let the difference between a coefficient and an exponent be carefully noted, and the office of each often explained; and illustrate frequently the signification of the language by attributing numerical values to letters in various algebraic expressions.

7. Point out often the characteristics of similar and dissimilar quantities, and explain which may be incorporated, and which cannot.

8. Explain the power of the minus sign, as shown in the four ground-rules; but very particularly, as it is illustrated in subtraction and multiplication.

9. Point out and illustrate the correspondence between the four ground-rules of Arithmetic and Algebra; and impress the fact, that their differences, wherever they appear, arise merely from differences in notation and language: the principles which govern the operations being the same in both.

10. Explain with great minuteness and particularity all the characteristic properties of the equation: the manner of forming it; the different kinds of quantity which enter into its composition; its exam-

ination or discussion; and the different methods of elimination.

11. In the equation of the second degree, be careful to dwell on the four forms which embrace all the cases, and illustrate by many examples that every equation of the second degree may be reduced to one or other of them. Explain very particularly the meaning of the term root; and then show, why every equation of the first degree has one, and every equation of the second degree two. Dwell on the properties of these roots in the equation of the second degree. Show why their sum, in all the forms, is equal to the coefficient of the second term, taken with a contrary sign; and why their product is equal to the absolute term with a contrary sign. Explain when and why the roots are imaginary.

12. In fine, remember that every operation and rule is based on a principle of science, and that an intelligible reason may be given for it. Find that reason, and impress it on the mind of your pupil in plain and simple language, and by familiar and appropriate illustrations. You will thus impress right habits of investigation and study, and he will grow in knowledge. The broad field of analytical investigation will be opened to his intellectual vision, and he will have made the first steps in that sublime science which discovers the laws of nature in

their most secret hiding-places, and follows them, as they reach out, in omnipotent power, to control the motions of matter through the entire regions of occupied space.

